A Novel Approach to Comparison-Based Diagnosis for Hypercube-Like Multiprocessor Systems

Chieh-Feng Chiang
cfchiang@cs.nctu.edu.tw

Jimmy J. M. Tan
jmtan@cs.nctu.edu.tw

Department of Computer Science, National Chiao Tung University, Hsinchu, Taiwan

Abstract

Interconnection networks have been an active research area for parallel and distributed computer systems. The diagnosability has played an important role in the reliability of interconnection networks. In this paper, we present a novel idea on system diagnosis called local diagnosability. There is a strong relationship between the local diagnosability and the traditional global one. For this local sense, we focus more on a single processor and require only identifying the status of this particular processor correctly. We propose a sufficient condition to determine the local diagnosability of a given processor. Moreover, we prove that the diagnosability of an n-dimensional hypercube-like network HL_n is n for $n \geq 5$ in this local sense, and show that the local diagnosability of each node in an n-dimensional hypercube-like network equals to its degree with up to $n-2$ faulty edges.

1 Introduction

With the rapid development of technology, multiprocessor systems are more and more important. The reliability of the processors in multiprocessor systems is therefore becoming an important issue. In order to maintain the reliability of a system, if a processor is found faulty, it should be replaced by a fault-free one. The procedure of identifying all the faulty nodes is called the diagnosis of the system. The maximum number of faulty nodes that the system can guarantee to identify is called the diagnosability of the system.

There are several approaches for interconnected processors to perform self-diagnosis for faulty ones. One major approach is called the comparison model, first proposed by Malek and Maeng [1] [2]. This approach performs the diagnosis by sending the same input to a pair of adjacent processors and comparing their responses.

In the previous studies on diagnosis, most investigators focused on the global diagnosis ability of a system but ignored some local systematic details. For example, if a system is of diagnosability t, it is at most t-diagnosable. That is, given any syndrome σ, all the faulty nodes in a system G can be precisely identified if G is with at most t faulty nodes. But it is possible to correctly point out all faulty nodes in some part of the system G under any given syndrome if G is with more than t faulty nodes. Thus, only considering the global status let us lose some local detail of a system.

In this paper, we present a novel idea on system diagnosis which is called local diagnosability. More local information about a system can be retrieved through this concept. In other words, every node in a system has its own local diagnosability which states some kind of connection status around it. Moreover, we proposed a sufficient condition to easily compute the local diagnosability of each node based on the comparison model. Finally, we can get back to the original global diagnosis in the point of view of local diagnosis. We prove that the diagnosability of an n-dimensional hypercube-like network HL_n is n for $n \geq 5$ in this local sense, and show that the local diagnosability of each node in an n-dimensional hypercube-like network equals to its degree with up to $n-2$ faulty edges.

2 Preliminaries

In this section, we give the basic of graph definition and notation [6]. $G = (V, E)$ is a graph if V is a finite set and E is a subset of $\{(u, v) \mid (u, v)$ is an unordered pair of $V\}$. The degree of node v in a graph G is the number of edges incident with v. A node cover of G is a subset $Q \subseteq V(G)$ such that every edge of $E(G)$ has at least one end node in Q. A node cover set with the minimum cardinality is called a minimum node cover.

Vaidya et al. [5] introduced a class of hypercube-like interconnection networks, called HL-graphs, which can be defined by applying the $\oplus$ operation repeatedly as follows: $HL_0 = \{K_1\}$; for $m \geq 1$,

$HL_m = \{G_0 \oplus G_1 \mid G_0, G_1 \in HL_{m-1}\}$, which has node set $V(G_0 \oplus G_1) = V(G_0) \cup V(G_1)$ and edge set $E(G_0 \oplus G_1) = E(G_0) \cup E(G_1) \cup M$, where M is an arbitrary perfect matching between the node set of G_0 and G_1. That is, M is a set of edges connecting the nodes of G_0 and G_1 in a bijection.

For the purpose of self-diagnosis of a given system, several different models have been proposed. The comparison model, called the MM model, proposed by Maeng and Malek [2] [3], is considered to be a major approach for fault diagnosis in multiprocessor systems. In this approach, for each processor w which has two distinct links to two other processors u and v, the diagnosis can be performed by sending two identical signals from w to u and from w to v, and then comparing their returning responses. Furthermore, in the *MM* model* [4], a special case of the MM model, it is assumed that a comparison is performed by each processor for each pair of distinct connected neighbors.

This diagnosis-by-comparison strategy can be modeled as a labeled multigraph $M = (V, C)$, called a comparison graph, where V represents the set of all processors in G and C represents the set of labeled edges. For each labeled edge $(u, v)_w \in C$, w is a label on the edge, which means that processors u and v are being compared by the *comparator* w. The output on a labeled edge $(u, v)_w \in C$ is denoted by $r((u, v)_w)$, which represent the comparison result of w for the two responses from u and v. An agreement is denoted by $r((u, v)_w) = 0$, whereas a disagreement is denoted by $r((u, v)_w) = 1$. If $r((u, v)_w) = 1$, at least one member of $\{u, v, w\}$ is faulty; or, if $r((u, v)_w) = 0$ and w is known to be fault-free, both u and v are fault-free.

For a given syndrome σ, a subset of nodes $F \subset V(G)$ is *consistent* with σ if the syndrome σ can be produced from the situation that all nodes in F are faulty and all nodes in $V - F$ are fault-free. Let σ_F denote the set of syndromes which are consistent with F. Notice that for a syndrome σ, there might be more than one faulty subset of V which are consistent with σ. A system is defined to be *diagnosable* if, for every syndrome σ, an unique set of nodes $F \subseteq V$ is consistent with it. In addition, a system is called *t-diagnosable* if the system is diagnosable as long as the number of faulty processors is at most t. The maximum number t for a system to be t-diagnosable is called the *diagnosability* of the system. Two distinct subsets of nodes $F_1, F_2 \subset V$ are *distinguishable* if and only if $\sigma_{F_1} \cap \sigma_{F_2} = \phi$; otherwise, F_1 and F_2 are *indistinguishable.*

Let $G = (V, E)$ be a graph and let $M = (V, C)$ be the comparison graph of G. Define the *order graph* [4] of a node $u \in V$ to be $G_u = (X_u, Y_u)$, where $X_u = \{v \mid \text{either } (u, v) \in E \text{ or } (u, v)_w \in C \text{ for some } w\}$ and $Y_u = \{(v, w) \mid v, w \in X_u \text{ and } (u, v)_w \in C\}$. For a given node $u \in V$, the *order* of u is defined as the cardinality of a minimum node cover of G_u. For a subset of nodes $U \subset V$, define $T(G, U)$ to be the set $\{v \mid (u, v)_w \in C \text{ and } u, w \in U \text{ and } v \in V - U\}$.

The following is a lemma presented by Sengupta and Dahbura to characterize whether a system is t-diagnosable.

Lemma 1. *[4] For every two distinct subsets of nodes F_1 and F_2, (F_1, F_2) is a distinguishable pair if and only if at least one of the following conditions is satisfied:*
1) $\exists u, w \in V - F_1 - F_2$ and $\exists v \in (F_1 - F_2) \cup (F_2 - F_1)$ such that $(u, v)_w \in C$,
2) $\exists u, v \in F_1 - F_2$ and $\exists w \in V - F_1 - F_2$ such that $(u, v)_w \in C$, or
3) $\exists u, v \in F_2 - F_1$ and $\exists w \in V - F_1 - F_2$ such that $(u, v)_w \in C$.

In the rest of this paper, we present our novel concept of local diagnosability under the comparison diagnosis model and discuss some properties of it.

3 Local diagnosability

In this section, we will define the definition of local diagnosability, and we will provide some practical theorems about local diagnosability. By these theorems, we can easily check the diagnosability of a system.

We now introduce the concept of a system being *locally t-diagnosable* at a given node.

Definition 1. *A system $G(V, E)$ is locally t-diagnosable at node $x \in V(G)$ if, given a test syndrome σ_F produced by the system under the presence of a set of faulty nodes F containing node x with $|F| \leq t$, every set of faulty nodes F' consistent with σ_F and $|F'| \leq t$, must also contain node x.*

An equivalent way of stating the above definition is given below.

Proposition 1. *A system $G(V, E)$ is locally t-diagnosable at node $x \in V(G)$ if, for each pair of distinct sets $F_1, F_2 \subset V(G)$ such that $F_1 \neq F_2$, $|F_1|, |F_2| \leq t$, and $x \in (F_1 - F_2) \cup (F_2 - F_1)$, (F_1, F_2) is a distinguishable pair.*

Then, we define the *local diagnosability* of a given node as follows.

Definition 2. *The local diagnosability $t_l(x)$ of a node $x \in V(G)$ in a system $G(V,E)$ is defined to be the maximum number of t for G being locally t-diagnosable at x, that is,*

$$t_l(x) = max\{t \mid G \text{ is locally } t - diagnosable \text{ at } x\}.$$

The concept of a system being *locally t-diagnosable* at a node x is consistent with the traditional concept of a system being *t-diagnosable* in the global sense. The relationship between these two is as follows.

Theorem 1. *A system $G(V,E)$ is t-diagnosable if and only if G is locally t-diagnosable at x, for every $x \in V(G)$.*

Theorem 2. *The diagnosability of a system $G(V,E)$ is t if and only if*

$$min\{t_l(x) \mid \forall x \in V(G)\} = t.$$

Now, we need some definitions for further discussion. For any set of nodes $U \subseteq V(G)$, $G[U]$ denotes the subgraph of G induced by the node subset U. Let H be a subgraph of G and v be a node in H, $deg_H(v)$ denotes the degree of v in subgraph H. For a given set of nodes $S \subseteq V(G)$, we use $G - S$ to denote the induced subgraph $G[V(G) - S]$. Let S be a set of nodes and x be a node *not* in S, we use $C_{x,S}$ to denote the connected component which x belongs to in $G - S$. The *symmetric difference* of two sets A and B is defined as the set $A \Delta B = (A \cup B) - (A \cap B)$.

In the following, we propose a sufficient condition for verifying whether a system G is locally t-diagnosable at a given node x.

Theorem 3. *A system $G(V,E)$ is locally t-diagnosable at a given node $x \in V(G)$ if, for every set of nodes $S \subset V(G)$, $|S| = p$, $0 \leq p \leq t-1$, and $x \notin S$, the cardinality of every node cover including x of the component $C_{x,S}$ is at least $2(t-p)+1$.*

Proof. We prove this theorem by contradiction. Suppose G is not locally t-diagnosable at node x. By Proposition 1, there exists two distinct sets of nodes $F_1 \neq F_2 \subset V$ with $|F_i| \leq t$, $i = 1, 2$, and $x \in F_1 \Delta F_2$, such that (F_1, F_2) is an indistinguishable pair. Let $S = F_1 \cap F_2$, then $|S| = p$, $0 \leq p \leq t-1$. According to the condition, the cardinality of a node cover including x of the component $C_{x,S}$ is at least $2(t-p)+1$. Since $|F_1 \Delta F_2| \leq 2(t-p)$ and $x \in F_1 \Delta F_2$, there is at least one node which is a member of the node cover of $C_{x,S}$ lying in $C_{x,S} - F_1 \Delta F_2$. Consequently, there is at least one edge of $C_{x,S}$ lying in $C_{x,S} - F_1 \Delta F_2$. Then, (F_1, F_2) is a distinguishable pair since it satisfies condition 1 of Lemma 1. Therefore G is locally t-diagnosable at node x by Proposition 1, which is a contradiction. □

We now propose a substructure at node x, called an extended star, which can guarantee the local diagnosability of node x.

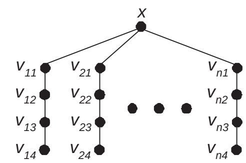

Figure 1: extended star sturcture $ES(x;n)$.

Definition 3. *Let x be a node in a graph $G(V,E)$. For $n \leq deg_G(x)$, an extended star $ES(x;n)$ of order n at node x is defined as $ES(x;n) = (V(x;n), E(x;n))$, where the set of nodes $V(x;n) = \{x\} \cup \{v_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq 4\}$ and the set of edges $E(x;n) = \{(x, v_{k1}), (v_{k1}, v_{k2}), (v_{k2}, v_{k3}), (v_{k3}, v_{k4}) \mid 1 \leq k \leq n\}$.*

We say that there is an *extended star* structure $ES(x;n) \subseteq G$ at node x if G contains an extended star $ES(x;n)$ of order n at node x as a subgraph.

Theorem 4. *Let x be a node in a system $G(V,E)$. The local diagnosability of x is at least n if there exists an extended star $ES(x;n) \subseteq G$ at x.*

Proof. We use Theorem 3 to prove this result. First, we define $l_k = (v_{k1}, v_{k2}, v_{k3}, v_{k4})$ to be a quadruple of four consecutive nodes for any k, $1 \leq k \leq n$, with respect to $ES(x;n)$. We note that l_k is a path of length 3. Accordingly, the cardinality of a node cover of each l_k is at least 2. Let $S \subset V(G)$ be a set of nodes in G with $|S| = p$, $0 \leq p \leq n-1$, and $x \notin S$. After deleting S from $V(G)$, there are at least $(n - p)$ complete l_k's still remaining in $ES(x;n)$, where the word "complete" means that all v_{k1}, v_{k2}, v_{k3}, and v_{k4} of an l_k have not been deleted in $G - S$. Thus, the cardinality of a node cover including x of the connected component $C_{x,S}$ is at least $1 + 2(n - p)$. Therefore, the system G with an extended star $ES(x;n)$ is locally n-diagnosable at x by Theorem 3. By Definition 2, the local diagnosability of x is at least n, i.e. $t_l(x) \geq n$. □

Theorem 5. *Let x be a node in a system $G(V,E)$ with $deg_G(x) = n$. The local diagnosability of x is at most n.*

Proof. We prove this proposition by contradiction. Suppose on the contrary that the local diagnosability of x is $n+1$ or more, i.e. $t_l(x) \geq n+1$. Let the nodes adjacent to x be v_k, for all k, $1 \leq k \leq n$. Let F_1 be the set $\{x\} \cup \{v_k \mid k = 1 \ to \ n\}$ and F_2 be the set $\{v_k \mid k = 1 \ to \ n\}$. Then, (F_1, F_2) is not a distinguishable pair according to Lemma 1, which is a contradiction. Then, the proof is completed. □

By Theorem 4 and Theorem 5, we have the following result.

Theorem 6. *Let x be a node in a system $G(V,E)$ with $deg_G(x) = n$. The local diagnosability of x is n if there exists an extended star $ES(x;n) \subseteq G$ at x.*

4 Diagnosability of Hypercube-Like Network

Theorem 7. *The diagnosability of an n-dimensional hypercube-like network HL_n is n for $n \geq 5$.*

Proof. We prove this theorem by induction on n, the dimension of hypercube-like network HL_n.
Basis: We now prove that HL_5 is 5-diagnosable. Consider any node x in HL_5, we find that there is a subgraph $ES(x,5)$ at x. Hence node x in HL_5 is locally 5-diagnosable by Theorem 6. Because HL_5 is node symmetric, every node in HL_5 is locally 5-diagnosable. Hence HL_5 is 5-diagnosable by Theorem 1.
Hypothesis: Suppose the claim holds for HL_{n-1}.
Induction: Consider an n-dimensional hypercube-like network HL_n. We want to show that each node of HL_n has a subgraph $ES(x,n)$ at it. Consider any node x in HL_n, we can separate HL_n into two HL_{n-1}, denoted by G and H. Without loss of generality, we may assume that x is in G. By hypothesis, there is a subgraph $ES(x,n-1)$ in G. Consider the corresponding node x' in H, there is a subgraph $ES(x',n-1)$ in H. Hence there is a subgraph $ES(x,n)$ in HL_n. Therefore x is locally n-diagnosable by Theorem 6. And HL_n is node symmetric, each node in HL_n is locally n-diagnosable, hence HL_n is n-diagnosable by Theorem 1. □

Theorem 8. *If the local diagnosability of each node in HL_{n-1} with $n-3$ edge faults equals to its degree, then the local diagnosability of each node in HL_n with $n-2$ edge faults equals to its degree.*

Proof. First we explain why the local diagnosability of each node in HL_n does not equal to its degree if there are $n-1$ faulty edges. We give an example in Fig. 2. Suppose all edges incident with node b

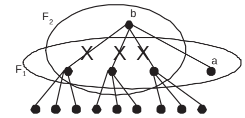

Figure 2: an indistinguishable pair.

exclusive $\{(a,b)\}$ are faulty. We can see that (F_1, F_2) is an indistinghushable pair by Lemma 1.

Now we are going to prove this theorem in the following. Suppose the local diagnosability of any node in HL_{n-1} with $n-3$ edge faults equals to its degree . Consider HL_n which has $n-2$ faulty edges is constructed with two copies of HL_{n-1}, one is G and the other is H. Without loss of generality, we may assume that x is in G and $deg(x) = m$. And the degree of node x' in H corresponding to x is m'. We prove it in two cases.

case **1**: There are k faulty edges that are crossed edge, where $1 \leq k \leq n-2$. See Fig. 3.

case **1.1**: Edge (x,x') is faulty.

Since there are k faulty edges in the crossed edge, where $1 \leq k \leq n-2$, the number of faulty edges in G is at most $n-3$. So the local diagnosability of x in G equals to its degree m. Hence there is a subgraph $ES(x,m)$ at x in HL_n because (x,x') is faulty. By Theorem 6, the local diagnosability of x in HL_n with $n-2$ faulty edges equals to its degree m .

case **1.2**: Edge (x,x') is fault-free.

Since there are k faulty edges in the crossed edge, where $1 \leq k \leq n-2$, the faulty edges in G and H are at most $n-3$. So the local diagnosability of x in G equal to its degree $m-1$ and the local diagnosability of x' in H equal to its degree $m'-1$. Hence there is a subgraph $ES(x,m-1)$ at x in G and $ES(x',m'-1)$ in H. So there is a subgraph $ES(x,m)$ at x in HL_n. By Theorem 6, the local diagnosability of x in HL_n with $n-2$ faulty edges equals to its degree m.

case **2**: There are no faulty edges that are crossed edge.

case **2.1**: All faulty edges are in G. (That is, there are $n-2$ faulty edges in G.) See Fig. 4.

If there is a faulty edge s belonging to $\{(x,v_{11}),(x,v_{21}),...,(x,v_{n1})\}$, we assume that s is fault-free. Hence there are $n-3$ faulty edges in G. By assumption, the local diagnosability of x in G equals to its degree. So we can find $ES(x,m-1)$ in G. Consider x' in H, we can also find $ES(x',m'-1)$ in H by assumption. Therefore, we can easily find $ES(x,m)$ in HL_n. Then the local diagnosability of x in HL_n with $n-2$ faulty edges equals to its degree by Theorem 6.

If there is a faulty edge s belonging to $\{(v_{11}, v_{12}), (v_{21}, v_{22}), ..., (v_{n1}, v_{n2})\}$, we assume that s is fault-free. Hence there are $n-3$ faulty edges in G. By assumption, the local diagnosability of x in G equals to its degree. So we can find $ES(x, m-1)$ in G. Consider x' in H, we can also find $ES(x', m'-1)$ in H by assumption. Consider node y' in H, we can find $ES(y', deg(y')-1)$ in H. Therefore, we can easily find $ES(x, m)$ in HL_n. Hence the local diagnosability of x in HL_n with $n-2$ faulty edges equals to its degree by Theorem 6.

If there is a faulty edge s belonging to $\{(v_{12}, v_{13}), (v_{22}, v_{23}), ..., (v_{n2}, v_{n3})\}$ or $\{(v_{13}, v_{14}), (v_{23}, v_{24}), ..., (v_{n3}, v_{n4})\}$, it can be proved by using the same way.

case **2.2**: There are k_1 faulty edges in G, where $1 \le k_1 \le n-2$. And there are k_2 faulty edges in H, where $1 \le k_2 \le n-2$. See Fig. 5.

Because the number of faulty edges in G and H is at most $n-2$. By the assumption, the local diagnosability of x in G equals to its degree, $m-1$. Hence we can find $ES(x, m-1)$ in G. We can find an $ES(x', m'-1)$ in H in the same way. Hence we can find an $ES(x, m)$ in HL_n. Therefore the local diagnosability of x in HL_n with $n-2$ faulty edges equals to its degree by Theorem 6.

In cases 1 and 2, we proved all possible distributions of faulty edges. Therefore, the proof is completed. □

5 Conclusions

The reliability of interconnection networks is an important issue. The diagnosability is also an important factor in measuring the reliability of interconnection networks. In this paper, we proposed a new point of view which is called the local diagnosability, and a theorem to verify the diagnosability of multiprocessor systems under the comparison-based diagnosis model. Then we prove the diagnosability of an n-dimensional hypercube-like network is n for $n \ge 5$, and show that the local diagnosability of each node in an n-dimensional hypercube-like network equals to its degree with up to $n-2$ faulty edges.

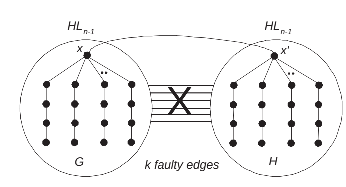

Figure 3: case1 of the proof in Theorem 8.

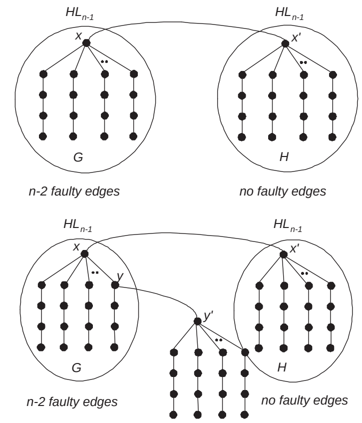

Figure 4: case2.1 of the proof in Theorem 8.

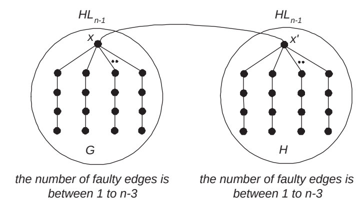

Figure 5: case2.2 of the proof in Theorem 8.

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