Proceedings of International Conference
on Computer Architecture

# On the Fault Diameter for Supercubes*

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## Abstract

*Assume that $N$ and $s$ are positive integers with $2^s < N \leq 2^{s+1}$. It is claimed by Auletta, Rescigno, and Scarano that the fault diameter of the supercube with $N$ nodes, is exactly $s+1$ if $N \notin \{2^{s+1}-1, 2^{s+1}-2, 2^s+2^{s-1}+1\}$, and $s+2$ otherwise. In this paper, we will argue that the above assertion is not correct. Instead, we will show that the fault diameter of the supercube with $N$ nodes is $s+1$ if $N \notin \{2^{s+1}-2\} \cup \{2^{s+1}-2^i+1 \mid 0 \leq i \leq s-1\}$, and $s+2$ otherwise. To get this goal, the shortest path routing algorithm for supercubes is also presented in this paper.*

## 1 Introduction and notations

Hypercube topology has been studied extensively as an interconnection network for parallel machines because of advantages like high bandwidth and low message latency [6]. One major constraint of the hypercube topology is that the number of nodes in the network must be $2^s$ for some positive integer $s$ and as such cannot be defined for any number of nodes. Incomplete hypercube topology proposed in [3] removed this restriction. However, the incomplete hypercube has serious limitations from the fault-tolerance perspective. A single node failure may disconnect the network. In [7], Sen proposed a family of networks, called supercubes and denoted by $S_N$. Each $S_N$ contains exactly $N$ nodes. If $N$ satisfies the relation $2^s < N \leq 2^{s+1}$, then $S_N$ is a supergraph of the hypercube with $2^s$ nodes. Later, much literature has investigated the topological properties of supercubes extending results known for the hypercube to the supercube [1,8,9]. This indicates that the performance of the supercube is almost the same as the hypercube which is about the same size. The fault diameter [4] is an important measure for interconnection networks. Assume that $2^s < N \leq 2^{s+1}$. It is claimed in [1] that the fault diameter of $S_N$ is exactly $s+1$ if $N \notin \{2^{s+1}-1, 2^{s+1}-2, 2^s+2^{s-1}+1\}$, and $s+2$ otherwise. In this paper, we will argue that the above assertion is not correct. Instead, we will show that the fault diameter of the supercube with $N$ node is $s+1$ if $N \notin \{2^{s+1}-2\} \cup \{2^{s+1}-2^i+1 \mid 0 \leq i \leq s-1\}$, and $s+2$ otherwise.

Now, we will formally introduce the definition of supercubes and some graph terminologies used in this paper. Most of the graph and interconnection network definitions used in this paper are standard (see e.g., [5]). Let $G = (V, E)$ be a finite, undirected graph. Throughout this paper, node and vertex are used interchangeably to represent the element of $V$. Edge and link are used interchangeably to represent the element of $E$. For a vertex $u$, $N(u)$ denotes the *neighborhood* of $u$ which is the set $\{v \mid (u,v) \in E\}$. Let $u, v$ be two nodes of $G$. The *distance* between $u$ and $v$, denoted by $d_G(u,v)$, is the length of the shortest path between them. The *diameter* of $G$, denoted by $D(G)$, is the maximum distance between any two nodes in $G$. The *connectivity* of $G$, denoted by $\kappa(G)$, is the minimum number of nodes whose removal leaves the remaining graph disconnected or trivial. Let $G = (V, E)$ be a graph with $\kappa(G) = \kappa$. It follows from Menger's theorem that there are $k$ *internal node-disjoint* (abbreviated as *disjoint*) *paths* joining any two vertices $u$ and $v$ when $k \leq \kappa$. Let $F$ be a subset of $V$ which is referred as a *faulty set*. $G - F$ denotes the subgraph induced by $V - F$. We use $d_k^f(G)$ to denote the largest diameter of $G - F$ for any faulty set $F$ with $|F| \leq k$. Obviously, $d_k^f(G) = \infty$ if $k \geq \kappa$. The *fault diameter* of a graph $G$ is defined as $d_{\kappa-1}^f(G)$. Obviously, we have $D(G) \leq d_{\kappa-1}^f(G)$.

Throughout this paper, we assume that $N$ and $s$ are positive integers with $2^s < N \leq 2^{s+1}$. Let $u = u_s u_{s-1} \dots u_1 u_0$ and $v = v_s v_{s-1} \dots v_1 v_0$ be two $(s+1)$-bit strings. The *Hamming distance* between

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$u$ and $v$, denoted by $h(u,v)$, is the number of $i$, $0 \le i \le s$, such that $u_i \ne v_i$. The $(s+1)$-*dimensional hypercube* consists of all the $(s+1)$-bit strings as its vertices and two vertices $u$ and $v$ are adjacent if and only if $h(u,v) = 1$. Hence each vertex of the $(s+1)$-dimensional hypercube is labelled with a unique integer $k$ with $0 \le k \le 2^{s+1} - 1$. Then the $N$-node supercube graph can be constructed from an $(s+1)$-dimensional hypercube as follows: For each node $u$ with $N \le u \le 2^{s+1} - 1$, merging nodes $u$ and $u - 2^s$ in the $(s+1)$-dimensional hypercube into a single node labeled as $u - 2^s$ and leaving other nodes in the $(s+1)$-dimensional hypercube unchanged, an $N$-node supercube is obtained.

More precisely, let $S_N = (V,E)$ be a *supercube*. The vertex set $V$ consists of $N$ vertices which are labeled from 0 to $N-1$. Then, each vertex $u$ ($0 \le u \le N-1$) can be expressed as an $(s+1)$-bit string $u_s u_{s-1} \dots u_1 u_0$ such that $u = \sum_{i=0}^{s} u_i 2^i$. In other words, an $(s+1)$-bit string $u_s u_{s-1} \dots u_0$ is a node of $S_N$ if and only if $u \le N-1$. Let $u = u_s u_{s-1} \dots u_1 u_0$ be an $(s+1)$-bit string. We use $\bar{u}$ to denote the string $\bar{u}_s \bar{u}_{s-1} \dots \bar{u}_1 \bar{u}_0$ and use $u^k$ to denote the string $u_s u_{s-1} \dots u_{k+1} \bar{u}_k u_{k-1} \dots u_0$. For $0 \le k \le s$, we may also use $u(k)$ to denote the bit $u_k$. The vertex set $V$ is partitioned into three subsets $V_1$, $V_2$, and $V_3$, where $V_3 = \{u \mid u \in V,\ u_s = 1\}$, $V_2 = \{u \mid u \in V,\ u_s = 0,$ and $u^s \notin V\}$, and $V_1 = \{u \mid u \in V,\ u_s = 0,$ and $u^s \in V\}$. The edge set $E$ is the union of $E_1$, $E_2$, $E_3$, and $E_4$, where $E_1 = \{(u,v) \mid u,v \in V_1 \cup V_2$ and $h(u,v) = 1\}$, $E_2 = \{(u,v) \mid u,v \in V_3$ and $h(u,v) = 1\}$, $E_3 = \{(u,v) \mid u \in V_3,\ v \in V_2$ and $h(u,v) = 2\}$, and $E_4 = \{(u,v) \mid u \in V_3,\ v \in V_1$, and $h(u,v) = 1\}$. As an example, a supercube with 12 nodes is shown in Figure 1. In this figure, edges in $E_1$, $E_2$, and $E_4$ are indicated by solid lines and edges in $E_3$ are indicated by dashed lines. Let $Z^0 = V_1 \cup V_2$ and $Z^1 = V_3$. Obviously, $Z^0$ induces an $s$-dimensional hypercube.

It is proved in [7] that $\kappa(S_N)$ is $s$ if $2^s < N < 2^s + 2^{s-1}$, and $s+1$ if $2^s + 2^{s-1} \le N \le 2^{s+1}$. In [1], it is claimed that the fault diameter $d^f_{\kappa-1}(S_N)$ of $S_N$ is $s+1$ if $N \notin \{2^{s+1}-1,\ 2^{s+1}-2,\ 2^s + 2^{s-1}+1\}$ and $s+2$ otherwise. However, this result is not true in general. For example, consider the case $S_{29}$ shown in Figure 2. The connectivity of $S_{29}$ is 5. Let $u = 01100$ and $v = 00011$ be two nodes of $S_{29}$. Assume that the faulty set of $S_{29}$ is $F = \{00100,\ 01000,\ 01110,\ 01101\}$ which is indicated by darkened nodes. Then use breadth first search rooted at $u$, $d_{S_N - F}(u,v) = 6$. Thus $d^f_{\kappa-1}(S_N) \ge 6$ and the result obtained in [1] is incorrect. In this paper, we will show that $d^f_{k-1}(S_N)$ is $s+1$ if $N \notin \{2^{s+1}-2\} \cup \{2^{s+1} - 2^i + 1 \mid 1 \le i \le s-1\}$, and $s+2$ otherwise. To get this goal, we will first present the shortest path routing algorithm for supercubes in the following section.

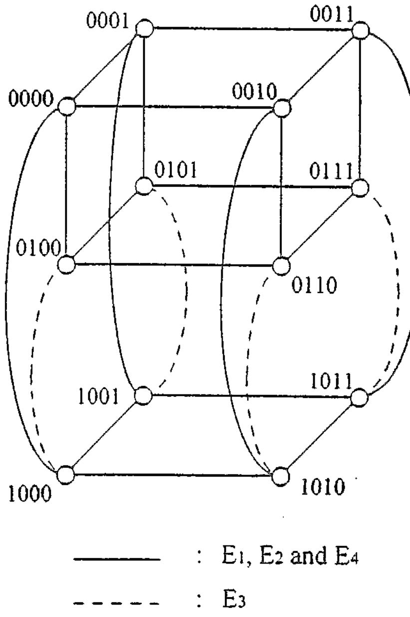


Fig.1 The supercube with 12 nodes

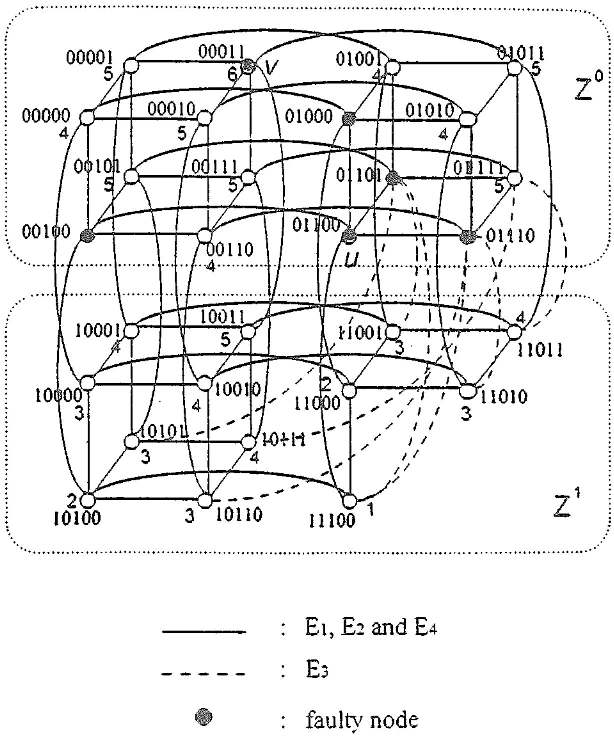


Fig.2 S29 with faulty set F={00100, 01000,01110, 01101}

## 2 Shortest path routing

Let $u$ and $v$ be two vertices of $S_N$ and $\{\alpha_i\}_{i=0}^{h(u,v)-1}$ be the decreasing sequence of the indices of binary representation such that $u_{\alpha_i} \neq v_{\alpha_i}$. Let $\vee$ denote the *string or* operator. For example, $11001 \vee 01011 = 11011$. The following lemma follows from the definition of a supercube.

**Lemma 1** *Assume that $u > v$. Let $P : x_0, x_1, x_2, \ldots, x_{h(u,v)}$ be a sequence of $(s+1)$-bit strings with $x_0 = u$ and $x_i = x_{i-1}^{\alpha_{i-1}}$ for $1 \leq i \leq h(u,v)$. Then $P$ forms a path joining $u$ to $v$ with all its internal nodes of $P$ less than $u$, i.e., $x_i \leq u$ for $1 \leq i \leq h(u,v) - 1$.*

**Lemma 2** *If both $u$ and $v$ are in $Z^0$, then $d_{S_N}(u,v) = h(u,v)$.*

**Proof.** It is observed that any edge $(x,y)$ in $E_1 \cup E_2 \cup E_4$ satisfies $h(x,y) = 1$, and any edge $(x,y)$ in $E_3$ satisfies $h(x,y) = 2$ and $x(s) \neq y(s)$, i.e, the $s$-th bit of $x$ is different from that of $y$. Since both $u$ and $v$ are in $Z^0$, $x(s) = y(s)$. It is observed that there are $h(u,v)$ bits, not including the $s$-th bit, that are different from $u$ to $v$. The length of any path joining $u$ and $v$ is at least $h(u,v)$. Without loss of generality, we assume that $u > v$. Applying Lemma 1, there is a path of length $h(u,v)$ joining $u$ to $v$. Hence $d_{S_N}(u,v) = h(u,v)$. □

**Lemma 3** *If both $u$ and $v$ are in $Z^1$, then $d_{S_N}(u,v) = h(u,v)$.*

**Proof.** As in Lemma 2, the length of any path joining $u$ and $v$ is at least $h(u,v)$. Without loss of generality, we assume that $u > v$. Applying Lemma 1, there is a path of length $h(u,v)$ joining $u$ to $v$. Hence $d_{S_N}(u,v) = h(u,v)$. □

**Lemma 4** *If $u$ is in $Z^1$ and $v$ is in $Z^0$, then $d_{S_N}(u,v)$ is $h(u,v)$ if $u \vee v < N$; and $h(u,v) - 1$ otherwise.*

**Proof.** It is observed that any edge $(x,y)$ in $E_1 \cup E_2 \cup E_4$ satisfies $h(x,y) = 1$, and any edge $(x,y)$ in $E_3$ satisfies (1) $h(x,y) = 2$, (2) $x(s) \neq y(s)$, and (3) $x \vee y \geq N$. The only available choice to decrease the routing distance is by taking an edge in $E_3$. Moreover, it is easy to see at most one $E_3$ edge, together with some of the other edges, is sufficient to construct the shortest path. In order to use any $E_3$ edge, we first have to route toward some node $x$ that is not in $S_N$, i.e., $x \geq N$. Then an edge in $E_3$ which is adjacent to $x^s$ can be chosen. Hence the lower bound for $d_{S_N}(u,v)$ is $h(u,v)$ if $u \vee v < N$, and $h(u,v) - 1$ otherwise. Suppose that $u \vee v < N$. It follows from Lemma 1 that we can construct a path $P$ of length $h(u,v)$ joining $u$ to $v$. Hence, $d_{S_N}(u,v) = h(u,v)$. Assume that $w = u \vee v \geq N$. Since both $u^s$ and $v$ are in $Z^0$ with $h(u^s, v) = h(u,v) - 1$, $d_{S_N}(u^s, v) = h(u,v) - 1$ follows from Lemma 2. Let $Q : u^s = x_0, x_1, \ldots, x_k = w^s, \ldots, x_{h(u,v)-1} = v$ be any shortest path joining $u^s$ to $v$ in $S_N$. Note that all nodes of $Q$ are in $Z^0$. Since $w \geq N$, there exists some $j$ with $1 < j \leq k$ such that $x_j^s \notin S_N$. Let $j$ be the smallest index with $x_j^s \notin S_N$. Then $(x_{j-1}^s, x_j)$ is an edge in $E_3$ with $x_{j-1}^s \in Z^1$ and $x_j \in Z^0$. Set $P$ as $u = x_0^s, x_1^s, \ldots, x_{j-1}^s, x_j, \ldots, x_{h(u,v)-1} = v$. Obviously, $P$ is a path of length $h(u,v) - 1$ joining $u$ and $v$ in $S_N$. Hence $d_{S_N}(u,v) = h(u,v) - 1$. □

We propose the shortest path routing algorithm in $S_N$ formally as follows:

*Routing algorithm*:
Let $u = u_s u_{s-1} \ldots u_1 u_0$ and $v = v_s v_{s-1} \ldots v_1 v_0$ be any two nodes in $S_N$ with $u > v$. Construct a shortest path $P$ from $u$ to $v$ as follows:

**Case 1** $u, v \in Z^0$ or $u, v \in Z^1$. Let $\{\alpha_i\}_{i=0}^{h(u,v)-1}$ be the decreasing sequence of indices such that $u_{\alpha_i} \neq v_{\alpha_i}$. A shortest path $P$ from $u$ to $v$ is constructed as: $x_0 = u, x_1, x_2, \ldots, x_{h(u,v)} = v$, where $x_i = x_{i-1}^{\alpha_{i-1}}$ for $1 \leq i \leq h(u,v)$.

**Case 2** $u \in Z^1$ and $v \in Z^0$. Set $w = u \vee v = 1w_{s-1}w_{s-2}\ldots w_0$. Let $\{\alpha_i\}_{i=0}^{h(u,w)-1}$ be the sequence of indices such that $u_{\alpha_i} \neq w_{\alpha_i}$ and $\{\beta_i\}_{i=0}^{h(w,v)-2}$ be the sequence of indices, not including $s$, such that $w_{\beta_i} \neq v_{\beta_i}$. Now we will construct the shortest path $P$ joining $u$ to $v$ as:

**Subcase 1** $u \vee v < N$. Set $P : x_0 = u, x_1, x_2, \ldots, x_{h(u,v)} = v$ as $x_{i+1} = x_i^{\alpha_{i+1}}$ for $0 \leq i \leq h(u,w) - 1$; $x_{h(u,w)+1} = x_{h(u,w)}^s$; and $x_i = x_{i-1}^{\beta_{i-h(u,w)-2}}$ for $h(u,w) + 2 \leq i \leq h(u,v)$.

**Subcase 2** $u \vee v \geq N$. Set $P : x_0 = u, x_1, x_2, \ldots, x_{h(u,v)-1} = v$ as follows: For $0 \leq i \leq h(u,w) - 1$, let $x_{i+1} = x_i^{\alpha_i}$ if $x_i^{\alpha_i} \in S_N$, and $x_{i+1} = (x_i^{\alpha_i})^s$ otherwise; and for $h(u,w) + 1 \leq i \leq h(u,v) - 1$, let $x_i = x_{i-1}^{\beta_{i-h(u,w)-1}}$.

## 3 Fault diameter of supercubes

To discuss the fault diameter of supercubes, we first consider the simpler case that $2^s < N < 2^s + 2^{s-1}$. In this case, $\kappa(S_N) = s$. Let $F$ be any faulty set with $|F| \leq s - 1$ and $u$, $v$ be any two nodes of $S_N - F$. Yuan [9] has shown that there are $s$ disjoint paths $P_0, P_1, \ldots, P_{s-1}$ in $S_N$ joining $u$ to $v$ such that the length of each path is at most $s + 1$. Obviously at least one of $P_i$ is fault-free; i.e., $P_i$ is in $S_N - F$. Thus $d_{\kappa-1}^f(S_N) \leq s + 1$. On the other hand, let $u = 1\overbrace{00\ldots0}^{s}$ and $v = 0\overbrace{11\ldots1}^{s}$ be two nodes in $S_N$. Let the faulty set $F = N(u) - \{u^s\}$. Hence $|F| = s - 1$. Let $Q : u = x_0, x_1, x_2, \ldots x_k = v$ be any path joining $u$ to $v$ in $S_N - F$. Obviously, $x_1 = u^s$. Thus, $d_{S_N-F}(u,v) = 1 + d_{S_N-F}(u^s, v)$. Since both $u^s$ and $v$ are in $Z^0$, $d_{S_N-F}(u,v) \geq 1 + d_{S_N}(u^s, v)$ $= 1 + h(u^s, v) = s + 1$ follows from Lemma 2. Thus, $d^f_{\kappa-1}(S_N) \geq s + 1$. We get the following theorem:

**Theorem 1** $d^f_{\kappa-1}(S_N) = s+1$ *if* $2^s < N < 2^s + 2^{s-1}$.

Now we will consider the case that $2^s + 2^{s-1} \leq N \leq 2^{s+1}$. In this case, $\kappa(S_N) = s+1$. In the remainder of this section, we assume that $2^s + 2^{s-1} \leq N \leq 2^{s+1}$.

**Lemma 5** $d^f_{\kappa-1}(S_N) \geq s+1$ *if* $2^s + 2^{s-1} \leq N \leq 2^{s+1}$. *Moreover,* $d^f_{\kappa-1}(S_N) \geq s + 2$ *if* $N \in \{2^{s+1} - 2\} \cup \{2^{s+1} - 2^i + 1 \mid 0 \leq i \leq s - 1\}$.

**Proof.** Let $u$ be the node which is labelled by $N - 1$ and $v = \overline{u}$. Hence $h(u,v) = s + 1$ and $h(u^s, v) = s$. Assume that $F = N(u^s) \cap Z^0$. Hence $|F| = s$ and $u^s, v \in Z^0$. Let $P : u^s = x_0, x_1, \ldots, x_k = v$ be any path in $S_N - F$ joining $u^s$ to $v$. Obviously, $x_1 = u$. By Lemma 4, $d_{S_N-F}(u,v) \geq h(u,v) - 1 = s$. Therefore, the length of $P$ is at least $s + 1$. Hence $d^f_{\kappa-1}(S_N) \geq s + 1$. Suppose that $N \in \{2^{s+1} - 2\} \cup \{2^{s+1} - 2^i + 1 \mid 0 \leq i \leq s-1\}$. Then $u = \overbrace{11\ldots1}^{s-1}01$ and $v = \overbrace{00\ldots0}^{s-1}10$ if $N = 2^{s+1} - 2$; and $u = \overbrace{11\ldots1}^{s-m+1}\overbrace{00\ldots0}^{m}$ and $v = \overbrace{00\ldots0}^{s-m+1}\overbrace{11\ldots1}^{m}$ with $0 \leq m \leq s - 2$ if $N \in \{2^{s+1} - 2^i + 1 \mid 0 \leq i \leq s-1\}$. Obviously, $x_2 = u^i$ for some $2 \leq i \leq s - 1$ if $N = 2^{s+1} - 2$; and $m \leq i \leq s - 1$ if $N \in \{2^{s+1} - 2^i + 1 \mid 0 \leq i \leq s-1\}$. Since $x_2 \vee v < N$ and $h(x_2, v) = s$, $d_{S_N-F}(x_2, v) = s$. The length of $P$ is at least $s + 2$. Therefore $d^f_{\kappa-1}(S_N) \geq s + 2$. □

The following theorem is proved in [6].

**Theorem 2** *For any two nodes $u$ and $v$ in the $n$-dimensional hypercube, there exist exactly $n$ disjoint paths joining $u$ to $v$; $h(u,v)$ of these paths are of length $h(u,v)$, and the remaining $n - h(u,v)$ paths are of length $h(u,v) + 2$.*

**Lemma 6** *Let $F$ be any faulty set with $|F| \leq s$. Then $d_{S_N-F}(u,v) \leq s + 1$ for any $u, v \in S_N - F$ with $h(u,v) = s + 1$.*

**Proof.** Let $u = u_s u_{s-1} \ldots u_0$ and $v = v_s v_{s_1} \ldots v_0$. We are going to construct $s+1$ disjoint paths, $P_0$, $P_1$, …, $P_s$, joining $u$ to $v$ such that the length of each path is at most $s + 1$. Then the proof of this lemma follows the construction. Since $h(u,v) = s+1$, $u = \overline{v}$. Without loss of generality, we may assume that $u_s$ is 0. Then both $u$ and $v^s$ are in $Z^0$. Since $Z^0$ induces an $s$-dimensional hypercube, it follows from Theorem 2 that there are $s$ internal node disjoint paths, $Q_0$, $Q_1, \ldots, Q_{s-1}$, in $Z^0$ joining $u$ to $v^s$ such that the length of each $Q_i$ is $s$. Note that $|N(v^s) \cap Z^0| = s$. We may write $N(v^s) \cap Z^0 = \{t_0, t_1, \ldots, t_s\}$ with $t_i = (v^s)^i$. Without loss of generality, we may assume that $t_i$ is in $Q_i$. Let $Q'_i$ be the subpath of $Q_i$ joining $u$ to $t_i$. Now we will construct $P_0$, $P_1$, …, $P_s$ as follows:

**Case 1** $u_{s-1} = 0$: Let $P_{s-1}$ be $u \overset{Q_{s-1}}{\to} v^s, v$, i.e., appending the edge $(v^s, v)$ to the path $Q_{s-1}$. For $0 \leq i \leq s - 2$, let $P_i$ be $u \overset{Q'_i}{\to} t_i, t^s_i, v$ if $t^s_i \in S_N$, and $u \overset{Q'_i}{\to} t_i, v$ otherwise. Then we set $P_s$ as $u, x_1 = u^s, x_2, \ldots, x_{s+1} = v$, where $x_i = x^{s-i}_{i-1}$ for $2 \leq i \leq s$ and $x_{s+1} = x^{s-1}_s$. It is observed that $x_k(s-1) = 0$ for $1 \leq k \leq s$. Thus $x_k < 2^s + 2^{s-1} \leq N$ for $1 \leq k \leq s$. All nodes in $P_s$ are in $S_N$. Note that $x_s = t^s_{s-1}$ and $x_s \notin P_i$ for $0 \leq i \leq s - 1$. It is easy to see that $P_0, P_1, \ldots, P_s$ are disjoint paths joining $u$ to $v$ such that the length of each path is at most $s + 1$.

**Case 2** $u_{s-1} = 1$: Let $P_0$ be $u \overset{Q_0}{\to} v^s, v$. For $1 \leq i \leq s - 1$, let $P_i$ be $u \overset{Q'_i}{\to} t_i, t^s_i, v$ if $t^s_i \in S_N$, and $u \overset{Q'_i}{\to} t_i, v$ otherwise. Then $t^s_0$ is not in any $P_i$ with $0 \leq i \leq s-1$. Let $x_1 = (u^s)^{s-1}$. Set the path $Q_s$ from $u$ to $x_1$ as $u$, $u^s$, $x_1$ if $u^s \in S_N$ and $u$, $x_1$ otherwise. Then set the sequence $Q'_s$ as $x_1, x_2, \ldots, x_s$, where $x_i = x^{s-i}_{i-1}$ for $2 \leq i \leq s$. Obviously, $x_{s-1} = t^s_0$ and $Q'_s$ forms a path from $x_1$ to $v$. Let $P_s$ be $u \overset{Q_s}{\to} x_1 \overset{Q'_s}{\to} v$. It is observed that $x_k(s-1) = 0$ for $1 \leq k \leq s - 1$. Thus $x_k < 2^s + 2^{s-1} \leq N$ for $1 \leq k \leq s$. Therefore, all nodes in $P_s$ are in $S_N$. Thus $P_s$ forms a path in $S_N$. It is easy to see that $P_0$, $P_1$, …, $P_s$ are disjoint paths joining $u$ to $v$ such that the length of each path is at most $s + 1$.

Hence, the lemma is proved. □

**Lemma 7** *Let $u = u_s u_{s-1} \ldots u_0$ be a node of $S_N$. $u_i = 0$ if $u^i \notin S_N$ for some $0 \leq i \leq s - 1$. In particular, $u = 11\overbrace{0\ldots0}^{s-1} = N - 1$ and $N = 2^s + 2^{s-1} + 1$ if $u^i \notin S_N$ for all $0 \leq i \leq s - 2$.*

**Proof.** The proof follows from the definition of supercubes. □

The following Lemmas are rather complicate. We omit the details of proof.

**Lemma 8** *Let $F$ be any faulty set of $S_N$ with $|F| \leq s$. Let $u$ and $v$ be any two nodes in $S_N - F$ with*

$h(u,v) \leq s-1$. *Then* $d_{S_N-F}(u,v) \leq s+2$ *if* $N \in \{2^{s+1}-2\} \cup \{2^{s+1}-2^i+1 \mid 0 \leq i \leq s-1\}$, *and* $d_{S_N-F}(u,v) \leq s+1$ *otherwise.*

**Proof.** Proof omitted. □

**Lemma 9** *Let* $F$ *be any faulty set of* $S_N$ *with* $|F| \leq s$ *and* $N \notin \{2^{s+1}-2\} \cup \{2^{s+1}-2^i+1 \mid 0 \leq i \leq s-1\}$. *Let* $u = u_s u_{s-1} \ldots u_0$ *and* $v = v_s v_{s_1} \ldots v_0$ *be any two nodes in* $S_N - F$ *with* $h(u,v) = s$. *Then* $d_{S_N-F}(u,v) \leq s+1$.

**Proof.** Proof omitted. □

**Lemma 10** *Let* $F$ *be any faulty set of* $S_N$ *with* $|F| \leq s$ *and* $N \in \{2^{s+1}-2\} \cup \{2^{s+1}-2^i+1 \mid 0 \leq i \leq s-1\}$. *Let* $u = u_s u_{s-1} \ldots u_0$ *and* $v = v_s v_{s_1} \ldots v_0$ *be any two nodes in* $S_N - F$ *with* $h(u,v) = s$. *Then* $d_{S_N-F}(u,v) \leq s+2$.

**Proof.** Proof omitted. □

It follows from Theorem 1 and Lemmas 5, 6, 8, 9, and 10, we get the following theorem.

**Theorem 3** $d^f_{\kappa-1}(S_N) = s+1$ *if* $N \notin \{2^{s+1}-2\} \cup \{2^{s+1}-2^i+1 \mid 0 \leq i \leq s-1\}$ *and* $d^f_{\kappa-1}(S_N) = s+2$ *otherwise.*

## 4 Concluding remarks

The performance of the supercube topologies is almost the same as the hypercube about the same size. In this paper, we show that the fault diameter of the supercube with $N$ nodes is $s+1$ if $N \notin \{2^{s+1}-2\} \cup \{2^{s+1}-2^i+1 \mid 0 \leq i \leq s-1\}$, and $s+2$ otherwise. Moreover, the shortest path routing algorithm for supercubes is also presented in this paper.

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