An Efficient Algorithm for Solving the Homogeneous Set Sandwich Problem

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Abstract

A set H of vertices of graph G(V, E) is a *homogeneous set* if each vertex in $V \setminus H$ is either adjacent to all vertices of H or to none of the vertices in H, where V and E are the vertex and edge, respectively, sets of graph G. A graph $G_s(V, E_s)$ is called a *sandwich graph* for the pair of graphs G(V, E) and $G_t(V, E_t)$ if $E_t \subseteq E_s \subseteq E$. The *homogeneous set sandwich problem* is to determine whether there exists a sandwich graph for the pair of graphs G and G_t such that there is a homogeneous set in G_s. In this paper, we shall propose an $O(n^3)$ time algorithm for solving the homogeneous set sandwich problem.

Keywords: Sandwich problem, Homogeneous set, Spanning subgraph, Strongly connected component, Algorithm.

1. Introduction

A set H of vertices in a graph G(V, E) is a *homogeneous set* if each vertex in $V \setminus H$ (i.e., the difference of two sets) is either adjacent to all vertices of H or to none of the vertices in H, where V and E are the vertex and edge, respectively, sets of graph G [4]. In order not to have a trivial definition, it is asked that the cardinality of H, denoted by $|H|$, is greater than or equal to 2 and $|V \setminus H| \geq 1$. A graph $G_t(V, E_t)$ is a *spanning subgraph* of G(V, E) if $E_t \subseteq E$. A graph $G_s(V, E_s)$ is called a *sandwich graph* for the pair of graphs G and G_t if $E_t \subseteq E_s \subseteq E$ [2]. The homogeneous set sandwich problem is to determine whether there exists a sandwich graph for the pair of graphs G and G_t such that there is a homogeneous set in G_s. For example, see Figure 1. Graphs Gs and G_t are spanning subgraphs of G. Graph G_s is a sandwich graph for the pair of graphs G and G_t. The vertex set {a, c, e} is a homogeneous set in G_s in which vertex d is adjacent to all of them and vertex b is adjacent to none of them. Note that {a, c, e} is not a homogeneous set in both of the graphs G and G_t.

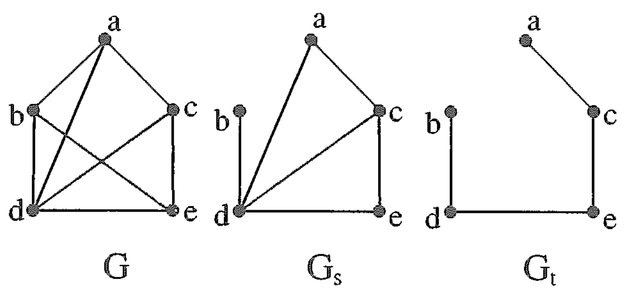

Figure 1. A sandwich graph which has a homogeneous set.

A homogeneous set can be found in polynomial time [5, 6, 7, 8]. In [8], Spinrad gave an $O(m\,\alpha(m,n))$ time algorithm to find a homogeneous set, where n and m are the number of vertices and edges, respectively, in a graph. The fastest algorithm proposed by McConnell and Spinrad takes O(m) time to find a homogeneous set. Cerioli et. al. Gave an $O(n^4)$ time algorithm for determining whether there exists a sandwich graph that admits a homogeneous [1]. In this paper, we shall propose an $O(n^3)$ time algorithm for solving the homogeneous set sandwich problem.

The remainder of this paper is organized as follows. In Section 2, we introduce Cerioli's Algorithm. In Section 3, we propose an efficient algorithm for solving the homogeneous set sandwich problem. Section 4 contains the concluding remarks.

2. Cerioli's Algorithm

In this section, we introduce the algorithm proposed by Cerioli et. al. [1]. Before describing their algorithm, we define some notation which will be used throughout this paper.

Let $G_s(V,E_s)$ be a sandwich graph of graphs G(V,E) and $G_t(V,E_t)$, where $E_t \subseteq E_s \subseteq E$. A vertex $u \in V$ is called a *bias vertex* of a vertex set $S \subset V \backslash \{u\}$ if there exists two vertices v and w in S such that there is an edge between u and v in E_t and no edge is between u and w in E. That is, u is partially adjacent to v and w in both G_t and G. A set C is called a *bias set* of vertex set S with respect to graphs G and G_t if C contains all bias vertices of S. For example, see Figure 1 again. Vertex e is the only bias vertex of vertex set {a, c}.

Now, we are at a position to describe Cerioli's Algorithm. We rewrite Cerioli's Algorithm as *Algorithm A*.

[Algorithm A]

[Input:] Two graphs G(V,E) and $G_t(V,E_t)$, where $E_t \subseteq E$.

[Output:] A sandwich graph $G_s(V,E_s)$ and a homogeneous set of G_s if it exists; otherwise, output no.

[Method:]

[Step 1.] For each pair of vertices x and y in G do

[Substep 1.] Let candidate homogeneous set H={x, y}

[Substep 2.] Find the bias set C of H.

[Substep 3.] If C is not an empty set, then let $H = H \cup C$ and go to Substep 2.

[Substep 4.] If $H \neq V$, then go to Step 3.

[Step 2.] There is no sandwich graph which admits a homogeneous set and stop.

[Step 3.] For every vertex $u \in V \setminus H$, if there is an edge between vertex u and one of the vertices in H, then add the edges between vertex u and all vertices in H to G_t. The resulting graph is G_s which admits a homogeneous set H.

[End of Algorithm A]

The most time-consuming step of Algorithm A is Substep 2. Intuitively, it needs $O(n^2)$ time to find a bias set C of H if H has O(n) vertices. Since there are $O(n^2)$ pairs of vertices in G, the time-complexity of *Algorithm A* is $O(n^4)$.

Theorem 1 [1] *Algorithm A correctly tests for the existence of a sandwich graph G_s that admits a homogeneous set H.*

3. An Efficient Algorithm to Find the Homogeneous Set of Some Sandwich graph

In this section, we shall present that the homogeneous set of a sandwich graph can be found in $O(n^3)$ time. The main idea of our algorithm is that we use the bias relation introduced in Section 2 to construct a directed graph called a b*ias graph* which will be defined later. Then, we can find the homogeneous set of a sandwich graph, if it exists, from the bias graph.

The *bias graph* $G_b(V_b, E_b)$ of a pair of graphs G(V,E) and $G_t(V,E_t)$ has vertex set $V \times V$, and there are two directed edges, i.e., *outgoing edges*, from vertex (u, v) to vertices (u, w) and (v, w) in G_b if and only if vertex w is a bias vertex of vertex set {u, v}. Notice that each vertex in G_b is composed of two different vertices in G, and vertex (x, y) and vertex (y,x) in G_b are combined into one vertex (x,y) with $x < y$. For example, the bias graph of the graphs G and G_t in Figure 1 is shown in Figure 2.

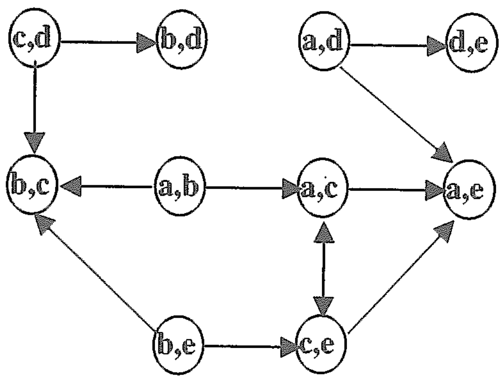

Figure 2. The bias graph of the graphs G and G_t in Figure 1.

A directed graph is *strongly connected* if there exists a path from u to v and also a path from v to u for every distinct pair of vertices u and v [3]. A *strongly connected component* of a directed graph is a maximal subgraph which is strongly connected. An *end strongly connected component* (*end component* for short) is a strongly connected component which does not have any outgoing edge to a vertex in another strongly connected component. In [9], Tarjan gave an O(n+m) time algorithm for finding the strongly connected components of a directed graph, where n and m are the number of vertices and directed edges, respectively, in a directed graph.

Now, we describe our $O(n^3)$ time algorithm for solving the homogeneous set sandwich problem as follows.

[Algorithm B]

[Input:] Two graphs G(V,E) and $G_t(V, E_t)$, where $E_t \subseteq E$.

[Output:] A sandwich graph $G_s(V, E_s)$ and a homogeneous set of G_s if it exists; otherwise, output no.

[Method:]

[Step 1.] Construct the bias graph G_b of graphs G and G_t.

[Step 2.] Find an end component of G_b. Let H denote the set of vertices in G which constitutes the vertices in the end component of G_b.

[Step 3.] If H=V, then there is no sandwich graph which admits a homogeneous set and stop.

[Step 4.] H is a homogeneous set of some sandwich graph G_s. By using the technique in Step 3 of *Algorithm A*, the corresponding sandwich graph G_s can be found.

[End of Algorithm B]

The vertices in G which form end components of the bias graph in Figure 2 are {b,d}, {d, e}, {b, c} and {a, e}. We can find that each of them is a homogeneous set of some sandwich graph. The Analysis of *Algorithm B* is as follows. Since each pair of vertices in G has at most O(n) bias vertices, each vertex in G_b has at most O(n) outgoing edges. Thus, in Step 1, Constructing the bias graph G_b takes $O(n^3)$ time. Notice that there are at most $O(n^3)$ directed edges in G_b. In Step 2, by applying the algorithm in [9], an end component of G_b can also be found in $O(n^3)$. Both of Steps 3 and 4 can be done in $O(n^2)$ time. Therefore, the time-complexity of *Algorithm B* is $O(n^3)$.

The correctness of *Algorithm B* is described by the following lemmas and theorem.

Lemma 2 *The graph $G_s(V, E_s)$ constructed by Algorithm B is a sandwich graph of graphs G(V,E) and $G_t(V, E_t)$.*

Proof : The construction of G_s in *Algorithm B* is described in Step 3 of *Algorithm A*. It can be found easily that $E_t \subseteq E_s \subseteq E$. Thus, G_s is a sandwich graph of graphs G and G_t. Q. E. D.

Lemma 3 *The set H of vertices found in Step 4 of Algorithm B is a homogeneous set of G_s.*

Proof : Since the bias set of H is empty in Step 4 of

Algorithm B, i.e., there is no bias vertex in V\H for the set H with respect to graphs G and G_t. Let u be a vertex in V\H. On one hand, if u does not have any edge incident to the vertices of H in G_t, then there may be no edge between u and the vertices of H in G_s. On the other hand, if u is incident to one of the vertices of H in G_t, then u must connect to all vertices of H in G and these edges may exist in E_s. Therefore, H is a homogeneous set of G_s.

Q. E. D.

Lemma 4 *If there exists a sandwich graph of graphs G and G_t which admits a homogeneous set, then Algorithm B will find a homogeneous set of some sandwich graph.*

Proof : Let G_b be the bias graph of graphs G and G_t. Since there exists a sandwich graph of graphs G and G_t which admits a homogeneous set, there must exist an end component in G_b. By Lemma 3, the vertices in G which constitute an end component of G_b form a homogeneous set of G_s. By applying Tarjan's algorithm [9], all connected components of G_b can be found. Thus, at least an end component can be found. This completes the proof.

Q. E. D.

We summarize Lemmas 2, 3 and 4 as the following theorem.

Theorem 5 *Algorithm B correctly tests for the existence of a sandwich graph G_s that admits a homogeneous set H in $O(n^3)$.*

4. Concluding Remarks

In this paper, we present an $O(n^3)$ time algorithm for finding a sandwich graph which admits a homogeneous set. Actually, with minor modification, our algorithm can find all homogeneous sets of some sandwich graphs. Whether there exists an O(n+m) time algorithm for solving the homogeneous set sandwich problem? It is an interesting open problem.

Another interesting problem is to discuss the number of edges in a bias graph. In our algorithm, we assume that there are $O(n^3)$ edges in a bias graph. Clearly, there is no bias edges in the bias graph of a complete graph and any one of its spanning subgraphs. We are studying it now.

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