Proceedings of International Conference
on Algorithms

Computing A Minimum Weight Triangulation Of A Spare Point Set [1]

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Abstract

Investigating the minimum weight triangulation of a point set with constraint is an important approach for seeking the ultimate solution of the minimum weight triangulation problem. In this paper, we consider the minimum weight triangulation of a spare point set, and present an $O(n^4)$ algorithm to computing a triangulation of such a set. The property of spare point set can be converted into a new sufficient condition for finding subgraphs of minimum weight triangulation. Special point set is exhibited to show that our new subgraph of minimum weight triangulation cannot be found by any currently known methods.

1 Introduction

Let $S = \{p_i \mid i = 0, ..., n-1\}$ be a set of n points in the plane, where each point p_i has the coordinates $(x(p_i), y(p_i))$. For simplicity, we assume that S is in general position so that no three points in S are co-linear. Let $\overline{p_ip_j}$ for $i \neq j$ denote the line segment with endpoints p_i and p_j, and let $\omega(p_ip_j)$ denote the weight of $\overline{p_ip_j}$, that is the Euclidean distance between p_i and p_j.

A *triangulation* of a planar point set S, denoted by $T(S)$, is a maximum set of line segments in which no two line segments share any interior point of them, thus $T(S)$ partitions the interior of the convex hull of S into empty triangles. The weight of a triangulation $T(S)$ is given by

$$\omega(T(S)) = \sum_{\overline{p_ip_j} \in T(S)} \omega(p_ip_j).$$

A *minimum weight triangulation*, simply MWT, of S is defined as

$$MWT(S) = min\{\omega(T(S)) \mid for\, all\, possible T(S)\}.$$

Computing an $MWT(S)$ is an outstanding open problem whose complexing status is unknown [GJ79]. A great deal of works has been done to seek the ultimate solution of the problem. Basically, there are two directions to attack the problem. The first one is to identify the edges inclusive or exclusive to $MWT(S)$ [Ke94, YXY94, CX96, DM96]. Xu [Xu92] showed that the intersection of all triangulations of S is a subset of $MWT(S)$. Recently, Dickerson and Montague [DM96] have shown that the intersection of all local optimal triangulations of S is a subgraph of $MWT(S)$. A triangulation $T(S)$ is called k-gon local optimal, denoted by $T_k(S)$, if any k-gon attracted from $T(S)$ is an optimal triangulation for this k-gon by the edges of $T(S)$. Then, the following inclusion property is hold:

$$\bigcap T(S) \subseteq \bigcap T_4(S) \subseteq \cdots \subseteq \bigcap T_{n-1}(S) \subseteq MWT(S).$$

However, it seems difficult to find the intersection as k increased, and only a subgraph of $T_4(S)$ has been found by [DM96]. Gilbert [Gi79] showed that the shortest edge in S is in $MWT(S)$. Yang, Xu, and You [YXY94] showed that mutual nearest neighbors in S are also in $MWT(S)$. Keil [Ke94] presented that the so-called β-skeleton of S for $\beta = \sqrt{2}$ is a subgraph of $MWT(S)$. Cheng and Xu [CX96] extended Keil's result to $\beta = 1.17682$. It seems that to identify more edges in $MWT(S)$ is a promising approach. This is because the more edges of $MWT(S)$ being identified, the less disjoint connected components in $MWT(S)$ existed. Thus, it is possible that eventually all these identified edges form a connected graph so that an $MWT(S)$ can be constructed by dynamic pro-

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gramming method in polynomial time. Moreover, even if such a connected graph is impossible to obtain, a larger subgraph will lead to the better performance of some heuristics [XZ96].

The other direction is to construct exact $MWT(S)$ with some constraint on S. Gilbert and Klinesek [Gi79,Kl80] investigated the case that S is restricted to a simple polygon. An $O(n^3)$ time dynamic programming algorithm was proposed to obtain an $MWT(S)$. Anagnostou and Corneil [AC93] studied the situation that S is restricted on k nested convex polygons. They gave an $O(n^{3k+1})$ time algorithm to find an $MWT(S)$. Meijer and Rappaport [MR92] later improved the time bound to $O(n^k)$ when S is restricted on k non-intersecting line segments inside the convex hull of S. Xu and Cheng etc. [Xu92, CGT95] showed that if a subgraph of $MWT(S)$ with k connected components is known, then the complete $MWT(S)$ can be computed in $O(n^{k+2})$ time. In addition to the potential applications of constraint cases, it is hope that the research on constraint cases would reveal some inside of the solution for general case.

In this paper, we investigate the situation that S forms a spare set, which informally speaking has a property that the distance between two consecutive convex layers of the set is longer than the diameter of the inner layer. We present an $O(n^4)$ time algorithm for computing an $MWT(S)$ for a spare set S. Amazingly, unlike the most known constrained MWT algorithms which are depended on the number of disjoint connected components, the time complexity of our algorithm is independent on the number of convex layers k. Furthermore, we can convert the property of spare set to a new sufficient condition for finding subgraphs of an $MWT(S)$. By demonstrating some special point set, we show that our new subgraphs cannot be found by any currently known methods [Gi79,Xu92,Ke94,CX96,DM96].

The paper is organised as follows. In Section 2, we discuss some properties of a point set restricted on its convex layers. In Section 3, we present an algorithm that produces an $MWT(S)$ with convex layers constraint. In Section 4, we define spare set S and propose an $O(n^4)$ algorithm to compute an $MWT(S)$. We further describe a sufficient condition for some edges to be in $MWT(S)$ and also demonstrate a point set whose MWT cannot be found by any known method. In Section 5, we make some concluding remarks.

2 Notations and Lemmas

The convex layers of a set S of point in the plane, denoted by $CL(S)$, is the set of nested convex polygons obtained by repeatedly computing the convex hull of the remaining set after removing the vertices of the current convex hull. Computing the convex layers of a planar point set was discussed in many papers [Ch85]. An optimal $\theta(n \log n)$ time algorithm was given by Chazelle.

Fact 1: *[Ch85]. Convex layers $CL(S)$ for $|S| = n$ can be found in $O(n \log n)$ time and $O(n)$ space.*

Let $CL(S) = (C_1, C_2, ..., C_k)$ be the convex layers of S, where C_i for $i = 1, ..., k$ is the ith layer of S. Let $V(C_i)$ be the vertex set of C_i and $R(C_i)$ be the interior region bounded by C_i. Assume that $|V(C_i)| = n_i$. The following relations hold.

$$\sum_{i=1}^{k} n_i = |S| \; and R(C_{i+1}) \subset R(C_i) \qquad (1)$$

Let $T_{CL}(S)$ be a triangulation of S with convex layers constraint, i.e., $CL(S) \subseteq T_{CL}(S)$. By Euler's formula, we have

$$|T_{CL}(S)| = 3n - |CH(S)| - 3 \qquad (2)$$

where $|T_{CL}(S)|$ denotes the size in terms of the edges and the above equality holds for any triangulation of S.

Lemma 1 *Let $CL(S) = (C_1, ..., C_k)$, where $|C_i| = n_i$ for $i \epsilon \{1, ..., k\}$ and let $R_{i,i+1} = R(C_i) - R(C_{i+1})$ for $i \epsilon \{1, ..., k-1\}$. Then, the number of edges of $T_{CL}(S)$ lying on $R_{i,i+1}$ is $n_i + n_{i+1}$.*

Proof By equations (1) and (2), the following equalities hold

$|T_{CL}(S)| = 3n - n_1 - 3$, and $|T_{CL}(S/V(C_1))| = 3(n - n_1) - n_2 - 3$.

The number of edges in $T_{CL}(S)$ lying on $R_{1,2}$ is $|T_{CL}(S)| - |T_{CL}(S/V(C_1))| - |CH(S)| = 3n - n_1 - 3 - (3(n - n_1) - n_2 - 3) - n_1 = n_1 + n_2$.

By applying the above analysis to any two consecutive convex layers, we can show that the number of edges in $T_{CL}(S)$ lying on $R_{i,i+1}$ is $n_i + n_{i+1}$, for any $i \epsilon \{1, ..., k-1\}$. □

Lemma 2 *Let $T(S)$ be any triangulation of S. The number of edges in $T(S)$ passing through the region $R_{i,i+1}$ is at least $n_i + n_{i+1}$ for $i \epsilon \{1, ..., k-1\}$.*

Proof If both C_i and C_{i+1} belong to $T(S)$, then by Lemma 1 the number of edges in $T(S)$ passing through $R_{i,i+1}$ is exactly $n_i + n_{i+1}$. Otherwise, some edges of $T(S)$ must cross $C_i \cup C_{i+1}$. Let $L_{i,i+1}$ be the subset of edges in $C_i \cup C_{i+1}$ and not in $T(S)$, and let $L^*_{i,i+1}$ be the subset of edges in $T(S)$ crossing some edges in $L_{i,i+1}$. Deleting $L^*_{i,i+1}$ from $T(S)$, adding $L_{i,i+1}$ to $T(S)/L^*_{i,i+1}$, and re-triangulating S with $(T(S)/L^*_{i,i+1}) \cup L_{i,i+1}$ constraint, we have a new triangulation $T^*(S)$ in which both C_i and C_{i+1} belong to $T^*(S)$. By equation (2), we have that

$$| T^*(S)/(T(S)/L^*_{i,i+1}) \cup L_{i,i+1} | + | L_{i,i+1} |$$
$$= | L^*_{i,i+1} | . \qquad (3)$$

Since the number of edges in $T^*(S)$ crossing $R_{i,i+1}$ is $n_i + n_{i+1}$, by equation (3) we have that $| L^*_{i,i+1} | \leq | L_{i,i+1} |$. Thus, the number of edges in $T(S)$ passing through $R_{i,i+1}$ is at least $n_i + n_{i+1}$. □

Note that Lemma 2 can also be proved by the matching theorem between triangulations in paper [AART95].

Lemma 3 *Let $CL(S) = (C_1, ..., C_k)$ and let $T_{CL}(S)$ be any triangulation with $CL(S)$ constraint. For each vertex p of C_i, There exists a vertex q of C_{i-1} such that edge $\overline{pq}$ is belong to $T_{CL}(S)$.*

Proof Let p be a vertex of C_i, $1 < i \leq k$. Since p is an interior point of $R(C_{i-1})$ and since the angle between any two consecutive edges with one endpoint p in $T_{CL}(S)$ must be less than π, there must exist an edge $e \epsilon T_{CL}(S)$ lying in $R_{i-1,i}$ such that p is an endpoint of e and the other endpoint of e is a vertex of C_{i-1}. □

Let $MWT_{CL}(S)$ denote the minimum weight triangulation of S with convex layers constraint.

3 The algorithm for computing an $MWT_{CL}(S)$

Let $T_{CL}(S)$ be any triangulation of S with $CL(S) \in T_{CL}(S)$, and let $\omega(T_{CL}(S))$ be its weight. A minimum weight triangulation with convex layers constraint, $MWT_{CL}(S)$, is one which minimizes $\omega(T_{CL}(S))$ among all possible $T_{CL}(S)$. It is obvious that to find an $MWT_{CL}(S)$ is easier than to find an $MWT(S)$. This is because the convex layers $CL(S)$ are already known to be a subset of $MWT_{CL}(S)$, a polynomial time algorithm for computing an $MWT_{CL}(S)$ is possible.

Fact 2: *[Li87] If L is a set of non-intersecting edges with endpoints in S such that $G(S, L)$ is a planar connected graph, then an MWT of S with L constraint, denoted by $MWT_L(S)$, can be found in $O(n^3)$ time for $| S |= n$.*

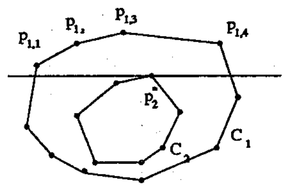

Figure 1:

Xu [Xu92] analyzed the optimal cell triangulation algorithm given by Heath and Pemmarajiu [HP92] and obtained an $O(n^3)$ algorithm for computing an $MWT_L(S)$, where L is a subset of non-intersecting edges with endpoints in S and $G(S, L)$ is a planar connected graph. We denote this algorithm as $\mathbf{A - T_L}$.

Since $MWT_{CL}(S)$ only minimizes the total weight of edges between convex layers, we first consider how to triangulate region $R_{1,2}$ so that the total weight of edges in $R_{1,2}$ is minimum. Let p_2^* be the vertex in C_2 with the maximum y-coordinate (for convenience, we can assume that no two points in S have a same y-coordinate), and let $N(p_2^*)$ be the subset of vertices of C_1 whose y-coordinates are greater than that of P_2^*, i.e., $y(p) > y(p_2^*)$ for $p \in N(P_2^*)$. Figure 1 shows the definition of p_2^* and $N(P_2^*)$, where $N(P_2^*) = (p_{1,1}, p_{1,2}, p_{1,3}, p_{1,4}, p_{1,5})$. By Lemma 3, there exists at least one point $p_1' \epsilon N(p_2^*)$ such that edge $\overline{p_2^* p_1'}$ is in an $MWT_{CL}(S)$. In order to identify such an edge, we have to check all possible edges ending at p_2^* and $N(p_2^*)$ and their corresponding constraint MWTs. Vertex p_2^* can be easily found in at most $O(|C_2|)$ time by scanning the y-coordinates of the vertices of C_2, $N(p_2^*)$ can be computed in at most $O(|C_1|)$ time by scanning the vertices of C_1 in upper half-plane above $y(p_2^*)$. For each point $p \epsilon N(p_2^*)$, add edge $\overline{pp_2^*}$ to form a graph $G(V(C_1) \cup V(C_2), C_1 \cup C_2 \cup \{\overline{pp_2^*}\})$. Clearly, the graph G is planar and connected. By Fact 2, an $MWT(V(C_1) \cup V(C_2))$ with $L(= C_1 \cup C_2 \cup \{\overline{pp_2^*}\})$ constraint can be found in $O((n_1+n_2)^3)$ time by algorithm $\mathbf{A-T_L}$. Then, an $MWT(V(C_1) \cup V(C_2))$ with $C_1 \cup C_2$ constraint can be found in at most $O(|N(p_2^*)|(n_1+n_2)^3)$ time.

In the following, we describe an algorithm, denoted by $A-MWT_{CL}$, to produce an MWT of S with convex layers constraint.

Let $CL(S) = (C_1, \cdots, C_k)$, and let p_i^* denote the vertex of C_i with maximal y-coordinate. Let $N(p_i^*)$ denote those vertices of C_{i-1} whose y-coordinates are greater than that of p_i^*.

ALGORITHM A-MWT$_{CL}$
Input: S (a set of n points in general position).
Output: $MWT_{CL}(S)$

1. Find the convex layers $CL(S) = (C_1, \cdots, C_k)$.
2. *For* $i = 2$ to k *Do*
   (a) Find p_i^* and $N(p_i^*)$.
   (b) *While* $N(p_i^*) \neq \emptyset$ *Do*
      i. $p \leftarrow attract(N(p^*))$;
      ii. Compute an $MWT_{C_i \cup C_{i-1} \cup \{\overline{p_i^* p}\}}$ $(V(C_i) \cup V(C_{i-1}))$ by $\mathbf{A-T_L}$;
      iii. Update $MWT_{C_i \cup C_{i-1}}$ $(V(C_i) \cup V(C_{i-1}))$
      iv. *EndWhile*
   (c) *EndDo*
3. produce $MWT_{CL}(S)$ by combining $MWT_{C_i \cup C_{i-1}}(V(C_i) \cup V(C_{i-1}))$ for all $i \epsilon [2, k]$.

The correctness and the time complexity of algorithm $\mathbf{A-MWT_{CL}}$ are shown as follows.

Theorem 1 *An $MWT_{CL}(S)$ can be found in $O(n^4)$ time, where S is a set of n points in general positions.*

Proof We apply $\mathbf{A-MWT_{CL}}$ to S, which correctly computes an $MWT_{CL}(S)$ since $\mathbf{A-T_L}$ correctly computes an $MWT_{C_i \cup C_{i-1} \cup \{\overline{p_i^* p}\}}$ $(V(C_i) \cup V(C_{i-1}))$. Consider the time complexity. Step 1 can be done in $O(n \log n)$ time by Fact 1 [Ch87]. Step 2 executed $k(= O(n))$ times, where Step (a) takes $O(n)$ time in the entire Step 2. By Fact 2, an $MWT_{C_1 \cup C_2}(R_{i,i-1})$ can be found in at must $O((n_i + n_{i-1})^3)$ time for $i = 2, \cdots, k$. Thus, Step (b) takes $O(n_i + n_{i-1})^3 * N(p_i^*))$ time. Since the process ends at finding an $MWT(R_{k-1,k})$, then the total computation in Step 2 is at most

$$\sum_{i=2}^{k} O(|N(p_i^*)|(n_i + n_{i-1})^3) \leq$$

$$(\sum_{i=2}^{k} |N(p_i^*))|O(n^3)$$

$$\leq O(n^4).$$

Step 3 takes $O(n)$ time. □

4 Computing an MWT of a Spare set

We now show that when S is a 'spare set', then $MWT_{CL}(S)$ is an $MWT(S)$. The **diameter** of a point set S, denoted by $D(S)$, is the maximum Euclidean distance among the pairs of points in S. The minimum **set distance** of two point sets S_1 and S_2, denoted by $d(S_1, S_2)$, is the minimum Euclidean distance between the points of S_1 and the points of S_2.

Let $CL(S)(= (C_1, \cdots, C_k))$ be the convex layers of a point set S. S is called **spare** if it satisfies the following two conditions:

(i) $d(V(C_i), V(C_{i+1})) \geq D(V(C_{i+1}))$, for all $i = 1, \cdots k-1$, and

(ii) if $\overline{p_i p_{i+1}}$ crosses $\overline{pq}$ for $\overline{p_i p_{i+1}} \in C_i$, $p, q \in S$, and $p \in C_j$ for $j < i$, then

$$d(p,q) > max\{d(p,p_i), d(p,p_{i+1})\}.$$

Theorem 2 *If S is a spare point set, then $CL(S) \subseteq MWT(S)$*

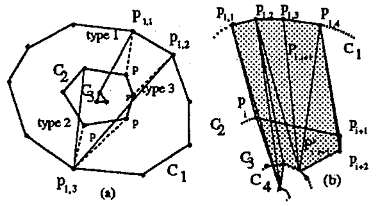

Figure 2:

Proof Let $CL(S) = (C_1, ..., C_k)$. Clearly, the convex hull of S, C_1, is in $MWT(S)$. We shall first prove that C_2 is in $MWT(S)$ by contradiction, that is suppose that there exists a subset of the edge set of C_2, say E, which does not belong to $MWT(S)$, then we can construct a new triangulation containing C_2 such that whose weight is less than that of $MWT(S)$. Let $\{\overline{p_1p_2}, \overline{p_2p_3}, \cdots, \overline{p_rp_{r+1}}\}$ be such a subset E, where the vertices $\{p_1, p_2, \cdots, p_{r+1}\}$ are in clockwise order around C_2. Let $\bar{E}$ be the set of edges in $MWT(S)$ such that each of which intersects an element of E. There are three types of edges in $\bar{E}$ as show in Figure 2(a). Deleting $\bar{E}$ from the edge set of $MWT(S)$ and adding E to $MWT(S)/\bar{E}$, we have $(MWT(S)/\bar{E}) \cup E$. For each edge $\overline{p_ip_{i+1}}$ of E, let $\bar{E}'$ be the subset of $\bar{E}$ crossing $\overline{p_ip_{i+1}}$. We connect all the endpoints of $\bar{E}'$ ending at C_1 to form a convex polygon $P_{i,i+1}$, that is, $P_{i,i+1} = (p_i, p_{i,1}, p_{i,2}, \cdots, p_{i,k_i}, p_{i+1}, p_i)$. In general, let these polygons be $P_{1,2} = (p_1, p_{1,1}, p_{1,2} \cdots, p_{1,k_1}, p_2, p_1)$; $P_{2,3} = (p_2, p_{2,1}, p_{2,2} \cdots, p_{2,k_2}, p_3, p_2); \cdots$; $P_{r,r+1} = (p_r, p_{r,1}, p_{r,2}, \cdots, p_{r,k_r}, p_{r+1}, p_r)$. Clearly, they are convex polygons lying outside C_2 and inside C_1. (See Figure 2(b), where $P_{i,i+1} = (p_i, p_{1,1}, p_{1,2}, p_{1,3}, p_{1,4}, p_{i+1}, p_i)$.) By Lemma 3, for every vertex p of C_2 there exists an edge $\overline{pp_{1,i}} \epsilon MWT(S)$ for $p_{1,i} \in C_1$ (as matter of a fact, this is true for any triangulation). Hence, all those convex polygons are disjoint due to the separation of these edges $\overline{pp_{1,i}}$. By connecting all these endpoints of $\bar{E}'$ lying below $\overline{p_ip_{i+1}}$ and inside $R(C_2)$, we determine a polygonal region, denoted by $P'_{i,i+1}$. (Refer to Figure 2 (b).) Let $d(p_{1,j}, q)$ denote the length of edge $\overline{p_{1,j}q}$ of $\bar{E}'$ i.e., the edge ending at $p_{1,j}$, and let $\omega(p_{1,j}) = d(p_{i,j}q)$. For each vertex on polygon $P_{i,i+1}$ for $1 \le i \le r$, $1 < j \le k_i$, we assign it a weight as follows.

(i) If $P_{i,i+1}$ contains only three vertices, i.e., $(p_i, p_{1,1}, p_{i+1})$, we assign p_i with $\omega(p_{1,1})$.

(ii) if $P_{i,i+1}$ contains more than three vertices, i.e., $(p_i, p_{1,1}, ..., p_{1,j}, ..., p_{1,k_1}, p_{i+1})$, we first assign p_i with $\omega(p_{1,1})$, then $p_{1,2}$ with $\omega(p_{1,2})$, ..., $p_{1,j}$ with $\omega(p_{1,j})$, ..., and $p_{1,k}$ with $\omega(p_{1,k_1})$.

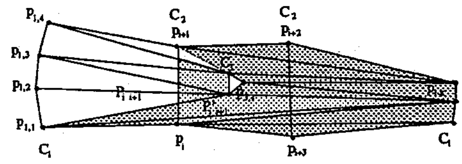

Figure 3: The light shaded area is $P'_{i,i+1}$ and the darker shaded area are $P_{i+2,i+3} \cup P'_{i+2,i+3}$.

It is clear that no two vertices matched a same edge since each vertex (except $p_{i,1}$ and p_{i+1}) is assigned a weight that equals to the length of an incident edge. In more detail, we consider three types of edges separately. It is obvious that a type 1 or a type 2 edge cannot be assigned to two different vertices in the sane convex layer. If the edge in question is of type 3, then both two vertices of this edge belong to the same convex layer, say C_1. However, since this edge crosses at least two convex layers and this edge must be shared by two triangles, there must exist more than one edge incidents at each of these two vertices. Thus, the two vertices can be assigned with two different edges. Refer to Figure 3, where $p_{1,1}$ and $p_{1,k}$ are such example. Hence, $p_{1,1}$ and $p_{1,k}$ can be assigned with different edges.

We re-triangulate each $P_{i,i+1}$, for $1 \le i \le r$, by adding edges $E_p^{*'} = \{\overline{p_ip_{i+1}}, \overline{p_ip_{i,j}}\}$ for $j = 2, \cdots, k_i$. Since $P_{i,i+1}$ is convex, the above re-triangulation is always possible. Let E_p^* be the set of such new edges in polygons $P_{1,2}, \cdots, P_{r,r+1}$. Thus, each of E_p^* is matched to a vertex in C_1 with an assigned weight. Thus, only the polygonal region $P'_{i,i+1}$ inside $R(C_2)$ remains to be triangulated. Note that the number of new edges needed to re-triangulate the interior of $P_{i,i+1} \cup P'_{i,i+1}$ is the same as $\bar{E}'$. By Lemma 2 and Lemma 3, we need add $\mid \bar{E}'/E_p^{*'} \mid$ new edges to triangulate $P'_{i,i+1}$. (In more detail, let us consider two cases: $\bar{E}'$ does not contain any type 3 edge and it contains some type 3 edges. In the former, the polygonal regions $P'_{i,i+1}$ are disjoint with different i, and the union of these regions is $R(C_2)$. In the latter, the two triangles sharing the type 3 edges are shared by the corresponding polygonal regions. For example in Figure 3, $P_{i,i+1} \cup P'_{i,i+1}$ and $P_{i+2,i+3} \cup P'_{i+2,i+3}$ share the triangle $\Delta p_{1,1}p_ip_{1,k}$ and $\Delta p_{1,1}p_{3,1}p_{1,k}$. Thus, the type 3 edge, $\overline{p_{1,1}p_{1,k}}$, must be counted only once in $P_{i,i+1} \cup P'_{i,i+1}$ for all $1 \le i \le r$.) Let all these new edges be denoted by E_p. Thus, the resulting triangulation will be $(MWT(S)/\bar{E}) \cup (E_p \cup E^*_p)$.

By inequality (ii) in the definition of spare set, $\omega(E^*_p)$ is less than the total weight of the assigned vertices, and by inequality (i), the weight of any edge in E_p is less than the weight of any edge in $\bar{E}$. So we have that $\omega(\bar{E}) > \omega(E_p \cup E^*_p)$, which contradicts the assumption that $MWT(S)$ is a minimum weight triangulation of S. Thus, $C_2 \in MWT(S)$ must hold.

By removing all the vertices of C_1, we have an original problem with one less convex layer. The above argument can be applied to $CL(S/V(C_1)) = (C_2, ..., C_k)$ again, so that $C_2 \in MWT(S)$ must hold. This proof continues until $CL(S/V(C_1) \cup, ..., \cup V(C_{k-1})) = C_k$. Then, $C_k \in MWT(S)$ must hold. □

In general speaking, $MWT_{CL}(S)$ is not an $MWT(S)$. But from Theorem 1 and Theorem 2, we have that

Theorem 3 *If S is a spare point set, then $MWT_{CL}(S) = MWT(S)$ and the $MWT(S)$ can be computed in $O(n^4)$ times.*

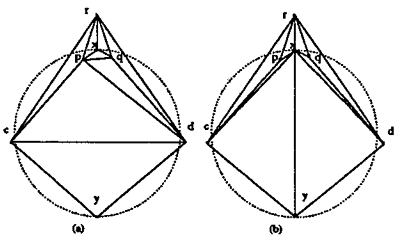

Figure 4:

By the analysis of computing an $MWT(S)$ of a spare set S, we can derive a sufficient condition for new subgraphs of MWT.

Sufficeint Condition

Let $CL(S) = (C_1, C_2, ..., C_k)$ be the convex layers of a point set S. Convex layer C_i for $1 < i \le k$ belongs to an $MWT(S)$ if the following conditions are satisfied:

(i) $d(V(C_s), V(C_{s+1})) \ge D(V(C_{s+1}))$, for all $s = 1, \cdots i-1$, and

(ii) if $\overline{p_sp_{s+1}}$ crosses $\overline{pq}$ for $\overline{p_sp_{s+1}} \in C_s$, $p, q \in S$, and $p \in C_j$ for $1 \le j < s \le i-1$, then

$$d(p,q) > max\{d(p,p_s), d(p,p_{s+1})\}.$$

The new subgraph (if it exists) is totally different from the known subgraphs given in [CX96, Gi79, Ke94, DM96, YXY94]. Figure 4(a) gives an example showing that our new subgraph is different from all the known subgraphs of [Gi79, Ke94, YXY94], where $\overline{pq}$ can be found by our method but $\overline{pq}$ does not belong to the subgraphs identified by any other method mentioned above. Clearly, x lies inside the empty disk associated with $\overline{pq}$ in Keil's β−skeleton and x also lies inside the empty double-circle in the condition of [YXY94]. $\overline{pq}$ is not the shortest edge among the seven points, thus, it cannot be found according to [Gi79]. $\overline{pq}$ is not a stable edge. Figure 4(b) shows that $\overline{pq}$ cannot be in $T_4(S)$ of [DM96] since $\overline{xy}$ belongs to a local optimal triangulation as shown.

5 Concluding Remarks

In this paper, we presented an $O(n^4)$ algorithm for computing an $MWT(S)$ of spare set S with n elements. We may regard that putting some constraint on point set S or designating some particular edges that must belong to $MWT(S)$ is a natural extension of $MWT(S)$ for a general point set S. In the latter, forcing the boundary of a simple polygon P to be in any $MWT(V(P))$ is a well-known constraint [Kl80]. Convex-layers constraint seems to be a reasonable extension with potential applications in this direction. It is quite interesting to find other constraints for MWT. In the former, restricting point set S to be on k convex layers [AC93] or to be on k non-intersecting straight line segments in $CH(S)$ [MR92] is this type of constraints. Spare set becomes another example.

The subgraph identified by our sufficient condition in section 4 is different from all the known subgraphs. It is interesting to see some experiment result based on our result.

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