Hamiltonian Problems on Ptolemaic Graphs *

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Abstract

This paper gives a unified approach for solving the Hamiltonian path, the Hamiltonian cycle problems and their variants on Ptolemaic graphs. These algorithms run in linear time.

1 Introduction

All graphs in this paper are finite, undirected, without loops or multiple edges. Let $G = (V, E)$ be a graph with $|V| = n$ and $|E| = m$. A connected graph is called *Ptolemaic* if and only if for any four vertices x, y, z, w of it we have the Ptolemaic inequality $d(x, y)d(z, w) \leq d(x, z)d(y, w) + d(x, w)d(y, z)$. Properties and optimization problems of Ptolemaic graphs have been studied in [2, 9, 15, 16, 20, 23, 18]. These graphs are superclasses of block graphs and subclasses of distance-hereditary graphs. A *Hamiltonian path* (respectively, *cycle*) of a graph G is a simple path (respectively, cycle) containing all vertices of G. The *Hamiltonian path* (respectively, *cycle*) *problem* is to determine whether a graph G has a Hamiltonian path (respectively, cycle) or not. This two problems are NP-complete for general graphs [13]. We will use the notations HC, HP, (HP, s) and (HP, s, t) as abbreviations for "Hamiltonian cycle", "Hamiltonian path", "Hamiltonian path with endpoint s" and "Hamiltonian path with endpoints s and t", respectively. Nicolai [20] presented the first polynomial-time algorithms for determining whether or not a Ptolemaic graph has an HP in $O(n^2(n+m))$ time, an (HP, s) in $O(n(n+m))$ time, an (HP, s, t) in $O(n + m)$ time, an HC in $O(n + m)$ time, for any s, $t \in V$, provided a *d-extremal dismantling scheme* is given (see [20] for definition). Whereas computing a d-extremal dismantling scheme requires $O(n^2)$ time [20] in a general graph. Though, most recently Dragan and Nicolai [11] presented a linear-time algorithm for computing such a d-extremal dismantling scheme on Ptolemaic graphs. They first gave an algorithm for the Hamiltonian cycle problem and then solved the Hamiltonian path problems by reducing them to the Hamiltonian cycle problem. It seems that exists a more efficient and unified algorithm for these Hamiltonian problems on Ptolemaic graphs if we avoid the use of d-extremal dismantling scheme and exploit the structure of Ptolemaic graphs. In this paper we give a unified approach to determine whether or not a Ptolemaic graph G has a HC, HP, (HP, s) or (HP, s, t) simultaneously in linear time. Our algorithm sharpens Nicolai's idea and does not use the d-extremal dismantling scheme. Notice that a graph G is Ptolemaic if and only if it is distance-hereditary and chordal [2, 9, 15, 16]. A graph is *distance-hereditary* if and only if every two vertices have the same distance in

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every connected induced subgraph [2, 9, 14]. Distance-hereditary graphs have been studied in [1, 2, 4, 5, 9, 10, 12, 14, 15, 16, 19, 21, 22, 23, 24, 6, 7]. A graph is *chordal* if every cycle of length > 3 has a chord. A graph is a *cograph* if there is no induced path containing 4 vertices.

2 Preliminaries

In this paper the terminology and notation of Bondy and Murty [3] are followed. Suppose A and B are two sets of vertices in a graph $G = (V, E)$. $G[A]$ denotes the subgraph of G induced by A. The *neighborhood* $N_A(B)$ of B in A is the set of vertices in A that are adjacent to some vertex in B. The *closed neighborhood* $N_A[B]$ of B in A is $N_A[B] \bigcup B$. For simplicity, $N_A(v)$, $N_A[v]$, $N(B)$, and $N[B]$ stand for $N_A(\{v\})$, $N_A[\{v\}]$, $N_V(B)$, and $N_V[B]$, respectively. The *distance* $d_G(x, y)$ or $d(x, y)$ between two vertices x and y in G is the minimum length of an x-y path in G. The *hanging* h_u of a connected graph $G = (V, E)$ at a vertex $u \in V$ is the collection of sets $L_0(u)$, $L_1(u)$,..., $L_t(u)$ (or L_0, L_1,..., L_t if there is no ambiguity), where $t = \max_{v \in V} d_G(u, v)$ and $L_i(u) = \{v \in V\colon d_G(u, v) = i\}$ for $0 \le i \le t$. For any $1 \le i \le t$ and any vertex $v \in L_i$, let $N'(v) = N(v) \bigcap L_{i-1}$. A vertex $v \in L_i$ with $1 \le i \le t$ has a *minimal neighborhood* in L_{i-1} if $N'(w)$ is not a proper subset of $N'(v)$ for any $w \in L_i$. For two disjoint vertex subsets X, Y of a graph $G = (V, E)$, they are said to be *joint* if each vertex of X is adjacent to each vertex of Y.

Notice that a Ptolemaic graph is chordal and distance-hereditary. In the following theorem several characterizations of distance-hereditary graphs and Ptolemaic graphs are stated, which are useful to our algorithm

Theorem 1 [2, 9, 14] *Suppose* $h_u = (L_0, L_1, \ldots, L_t)$ *is a hanging of a connected distance-hereditary graph* G *at* u.
(1) *For any two vertices* x, $y \in L_i$, $1 \le i \le t$, *we have that* $N'(x)$ *and* $N'(y)$ *are either disjoint, or one of the two sets is contained in the other. Moreover, if* G *is Ptolemaic then every* $N'(x)$ *induces a complete graph of* G.
(2) *There exists a vertex* $v \in L_i$ *such that* v *has a minimal neighborhood in* L_{i-1}. *In addition, if* v *satisfies the above condition then for every pair of vertices* x *and* y *in* $N'(v)$, *we have* $N_{V-N'(v)}(x) = N_{V-N'(v)}(y)$.
(3) *For every pair of vertices* x, $y \in L_i$, $1 \le i \le t$, *that are in the same component of* $G[V - L_{i-1}]$, *we have* $N'(x) = N'(y)$.
(4) *Every* L_i *induces a cograph in* G.

The following observation is the base of our algorithms: In Theorem 1, suppose $t > 1$, let Y be a component of $G[L_t]$ with $|N_{L_{t-1}}(Y)| \le |N_{L_{t-1}}(B)|$ for every component B in $G[L_t]$. Let $X = N_{L_{t-1}}(Y)$ and $Z = N[X] - (X \cup Y)$. It is clear that X, Y and Z are disjoint sets with $N[Y] \subseteq (X \cup Y)$ and $N[X] \subseteq (X \cup Y \cup Z)$. ¿From Theorem 1(3) and the choice of Y we note that any vertex in Y has a minimal neighborhood in L_{t-1}, meanwhile X and Y are joint. Therefore by Theorem 1(2) X and Z are joint. Moreover X will induce a complete subgraph of G when G is a Ptolemaic graph.

3 Hamiltonian Problems

Throughout this section, X, Y, and Z will denote three nonempty disjoint vertex subsets of a graph $G = (V, E)$ with $N[Y] \subseteq (X \cup Y)$ and $N[X] \subseteq (X \cup Y \cup Z)$ such that X and Y are joint, X and Z are joint, and X induces a complete subgraph of G, Y induces a cograph. An *HP* (respectively, *HC*, (HP, s), (HP, s, t) with $|\{s, t\} \cap Y| \le 1$ is (X, Y)-*canonical* if it contains a subpath that visits all vertices in Y and no vertices in $V - (X \cup Y)$. An hamiltonian path $\mathcal{P}$ which is an (HP, s, t) having $\{s, t\} \subseteq Y$ is (X, Y)-*canonical* if $\mathcal{P} = P_1P_2P_3$ such that (i) P_1 starts from s, (ii) P_3 ends at t, (iii) P_1 and P_3 do not visit any vertex in $V - (X \cup Y)$, (iv) P_2 does not visit any vertex in Y, and (v) either P_1 or P_3 has all its vertices in Y. For a graph H, $\pi_0(H)$ denotes the minimum number of pairwise disjoint paths covering H, $\pi_1(H, s)$ denotes the minimum number of pairwise disjoint paths covering H such that s is endpoint of one of these paths, and $\pi_2(H, s, t)$ denotes the minimum number of pairwise disjoint paths covering H such that s and t are endpoints of two of these paths, or $\pi_2(H, s, t) = 1$ if H contains an *HP* with endpoints s and t. For notational convenience we will use $\pi_0(Y)$, $\pi_1(Y, s)$ and $\pi_2(Y, s, t)$ to denote $\pi_0(G[Y])$, $\pi_1(G[Y], s)$ and $\pi_2(G[Y], s, t)$ respectively. We say that a subpath P' of a path P is an (X, Y)-*path* if P' starts from a vertex in X, ends at a vertex in Y, and has all its vertices in $X \cup Y$. A subpath P' of a path P is (X, Y)-*maximal* if P' is an (X, Y)-path and is not a proper subpath of any (X, Y)-path of P. For a subset W of vertices of a graph G, we say that a subpath P is a (W)-path if P has all its vertices in W. A subpath P' of a path P is (Y)-*maximal* if P' is a (Y)-path and is not a proper subpath of any (Y)-path of P. Suppose $P = p_1p_2 \cdots p_k$ is a path and p_i's are vertices visited by path P in the ordering that path P visits them. The *reverse path* of P, denoted by $\overline{P}$, is the path that visits vertices p_i's for all $1 \le i \le k$ in the reverse ordering that path P visits them.

Lemma 2 *If G has an (HP, s, t), then G has an (X, Y)-canonical (HP, s, t).*

Proof. Suppose $\mathcal{P}$ is an (HP, s, t) of G. If $\mathcal{P}$ has no (X, Y)-path, then it starts from vertex $s \in Y$ and visits all vertices in Y before it visits any vertex not in Y. Clearly, it is an (X, Y)-canonical (HP, s, t). Thus, we assume that $\mathcal{P}$ has an (X, Y)-path. Let $\{P_{xy}^i : 1 \leq i \leq k\}$ be the set of all (X, Y)-maximal subpaths of $\mathcal{P}$. Without loss of generality, assume that $t \in Y$ if $|Y \cap \{s, t\}| \geq 1$. Then, there are four cases:
Case 1, $\mathcal{P} = P_s P_{xy}^1 P_{xz}^1 P_{xy}^2 P_{xz}^2 \ldots P_{xz}^{k-1} P_{xy}^k P_t$,
Case 2, $\mathcal{P} = P_{xy}^1 P_{xz}^1 P_{xy}^2 P_{xz}^2 \ldots P_{xz}^{k-1} P_{xy}^k P_t$,
Case 3, $\mathcal{P} = P_s P_{xy}^1 P_{xz}^1 P_{xy}^2 P_{xz}^2 \ldots P_{xz}^{k-1} P_{xy}^k$, and
Case 4, $\mathcal{P} = P_{xy}^1 P_{xz}^1 P_{xy}^2 P_{xz}^2 \ldots P_{xz}^{k-1} P_{xy}^k$.

Clearly, each path P_{xz}^i starts from a vertex in X, ends at a vertex in Z, and has all its vertices in $V - Y$ for all i, $1 \leq i < k$.

Case 1. Since we assume that $t \in Y$ if $|Y \cap \{s, t\}| \geq 1$, therefore $s \notin Y$ in this case. Otherwise, $t \in Y$ and hence P_t will contain an (X, Y)-maximal path, a contradiction. Thus, both paths P_s and P_t do not visit any vertex in Y, P_s ends at a vertex in Z, and P_t starts from a vertex in X. Clearly the following path is an (X, Y)-canonical (HP, s, t):

$$P_s P_{xy}^1 P_{xy}^2 P_{xy}^3 \ldots P_{xy}^k P_{xz}^1 P_{xz}^2 \ldots P_{xz}^{k-1} P_t.$$

Case 2. This case can be proved by arguments similar to those for proving the above case.

Case 3. If $s \in Y$, then either P_s is a Y-path or $P_s = P_y P'$ where P_y is a Y-path, P' does not visit any vertex in Y and P' ends at a vertex in Z. In both cases, the following path is an (X, Y)-canonical (HP, s, t):

$$P_s P_{xz}^1 P_{xz}^2 \ldots P_{xz}^{k-1} P_{xy}^1 P_{xy}^2 P_{xy}^3 \ldots P_{xy}^k.$$

Case 4. Clearly, $P_{xy}^1 = P_x P$" where P_x is an X-path and P" starts from a vertex in Y. Let P' be the reverse path of the following path:

$$P_{xz}^1 P_{xz}^2 \ldots P_{xz}^{k-1}.$$

Obviously, P' starts from a vertex in Z and ends at vertex in X. Then, the following path is an (X, Y)-canonical (HP, s, t):

$$P_x P' P" P_{xy}^2 P_{xy}^3 \ldots P_{xy}^k.$$

∎

We use $P - t$ to denote the subpath P' of P such that $P = P't$ and t is the last vertex visited by P. We use $P - P'$ to denote the subpath P" of P such that $P = P"P'$. Let $\mathcal{P}$ be an (X, Y)-canonical (HP, s, t) of G. Suppose $\mathcal{P}$ has k (Y)-maximal paths. For $1 \leq i \leq k$, let P_x^i and P_y^i be (X)-maximal and (Y)-maximal path, respectively. For simplicity, we use $\mathcal{P} - (X' \cup Y)$ to denote the subpath of $\mathcal{P}$ obtained in all of the following three cases.

Case 1, both s and t are in Y.

In this case, without loss of generality, we assume that $\mathcal{P} = P_y^1 P_{xx} P_y^2 P_x^1 \cdots P_y^{k-1} P_x^{k-2} P_y^k$ where P_y^1 starts from vertex s, P_y^k ends at vertex t, P_{xx} starts from a vertex in X and ends at a vertex in X, does not visits any vertex in Y. Obviously one has $|X| \geq k \geq \pi_2(Y, s, t)$. We use $\mathcal{P} - Y$ to denote path $P_{xx} P_x^1 P_x^2 \cdots P_x^{k-2}$. Note that $\mathcal{P} - Y$ contains an (X)-path that visits at least $k - 1$ vertices of X. We use $\mathcal{P} - (X' \cup Y)$ to denote the path obtained by removing the last $|X'|$ vertices from path $\mathcal{P} - Y$. Since X and Z are joint, X and Y are joint, and X is a clique, we may assume that X' is any subset of X with $|X'|$ less than or equal to the number of vertices visited by path $P_x^1 P_x^2 \cdots P_x^{k-2}$. It is easy to see that $\mathcal{P} - (X' \cup Y)$ starts from a vertex in X and ends at a vertex in X.

Case 2, at most one of s and t is in Y.

Without loss of generality, assume that $t \in Y$. In this case,
$\mathcal{P} = P_w P_y^1 P_x^1 P_y^2 P_x^2 \cdots P_y^{k-1} P_x^{k-1} P_y^k$
where P_y^k ends at vertex t, P_w starts from s, ends at a vertex in X, does not visits any vertex in Y. Obviously, $k \geq \pi_1(Y, t)$ and $|X| \geq k \geq \pi_1(Y, t) \geq \pi_0(Y)$. In this case, we use $\mathcal{P} - Y$ to denote path $P_w P_x^1 P_x^2 \cdots P_x^{k-1}$. Note that $\mathcal{P} - Y$ contains an (X)-path that visits at least k vertices of X. We use $\mathcal{P} - (X' \cup Y)$ to denote the path obtained by removing the last $|X'|$ vertices from path $\mathcal{P} - Y$ with $|X'|$ less than or equal to the number of vertices visited by path $P_x^1 P_x^2 \cdots P_x^{k-1}$. That is, $\mathcal{P} - (X' \cup Y) = (\mathcal{P} - Y) - P_x^*$ where the set of vertices visited by P_x^* is X'.

Case 3, neither s nor t is in Y.

In this case,
$\mathcal{P} = P_w^1 P_y^1 P_x^1 P_y^2 P_x^2 \cdots P_y^{k-1} P_x^{k-1} P_y^k P_w^2$
where P_w^2 starts from a vertex in X, ends at vertex t, P_w^1 starts from s, ends at a vertex in X, neither P_w^1 nor P_w^2 visits any vertex in Y. Obviously, $k \geq \pi_0(Y)$ and $|X| > k \geq \pi_0(Y)$. In this case, we use $\mathcal{P} - Y$ to denote path $P_w^1 P_x^1 P_x^2 \cdots P_x^{k-1} P_w^2$. Let P_w^1 ends at x_1 and P_w^2 starts from x_2. Path $x_1 P_x^1 P_x^2 \cdots P_x^{k-1} x_2$ is an (X)-path that visits at least $k + 1$ vertices. Let X' be the last $|X'|$ vertices visited by path $x_1 P_x^1 P_x^2 \cdots P_x^{k-1} x_2$. We use $\mathcal{P} - (X' \cup Y)$ to denote the path obtained from $\mathcal{P} - Y$ by removing the last $|X'|$ vertices from path $x_1 P_x^1 P_x^2 \cdots P_x^{k-1} x_2$ where $|X'| \leq k$. That is,
$\mathcal{P} - (X' \cup Y) = (P_w^1 P_x^1 P_x^2 \cdots P_x^{k-1} - P_x^*) P_w^2$
where the set of vertices visited by P_x^* is X'.

The concept of $\mathcal{P}-(X'\cup Y)$ will be frequently used in the proof of lemmas for developing our algorithm.

Lemma 3 (1) *If G has an (HP,s), then G has an (HP,s) such that at most one of its endpoint is in Y.*
(2) *If G has an HP and $|X|>\pi_0(Y)$, then G has an HP such that neither of its two endpoints is in Y.*
(3) *If $|X|>\pi_0(Y)$, $s\notin Y$, and G has an (HP,s), then there exists an (HP,s) such that neither of its two endpoints is in Y.*

Proof. (1) Suppose $\mathcal{P}$ is an (HP,s) of G that starts from s and ends at t. If $t\notin Y$, then the lemma is true already. In the following, we assume tht both s and t are in Y. By Lemma 2, assume that $\mathcal{P}$ is (X,Y)-canonical and $\mathcal{P}=P_yP_{xz}P_{xy}$ where P_y, P_{xz}, and P_{xy} are subpaths of $\mathcal{P}$, P_y starts from s, P_{xy} ends at t, P_{xy} is an (X,Y)-maximal path, P_{xz} starts from a vertex in X, ends at a vertex in Z and does not visits any vertex in Y. Clearly, $P_yP_{xy}P_{xz}$ is an (HP,s) such that its endpoint other than s is not in Y.

(2) Suppose $\mathcal{P}$ is an HP of G such that $\mathcal{P}$ starts from s and ends at t. If neither s nor t is in Y, then this statement is true. By statement (1) of this lemma, we may assume that $\mathcal{P}$ is an HP of G such that one of its endpoint is in Y and the other of its endpoint is not in Y. Without loss of generality, assume that $t\in Y$ and $s\notin Y$. For notational convienence, let $k=\pi_0(Y)$. By Lemma 2, we may assume that $\mathcal{P}$ is an (X,Y)-canonical HP. Thus, $\mathcal{P}=P_sP_{xy}$ where P_{xy} is an (X,Y)-maximal path. Since $Z\neq\emptyset$, P_s ends at a vertex $z\in Z$. Consider graph $G[P_{xy}]$ which is the subgraph of G induced by the vertices visited by path P_{xy}. Suppose P_s visits a vertex in X. If $s\in X$, then $P_{xy}P_s$ in HP that starts from a vertex in X and ends at a vertex in Z. Otherwise, let $P_s=P_s^1P_s^2$ where P_s^1 ends at vertex in Z and P_s^2 starts from a vertex in X and ends at a vertex in Z. Then, $P_s^1P_{xy}P_s^2$ is an HP whose endpoints are not in Y. In the following, we assume that P_s does not visits any vertex in X. That is, P_{xy} is an HP of $G[X\cup Y]$. Since $|X|>\pi_0(Y)$, there exists an HP, P', of $G[X\cup Y]$ such that $P'=P_xP_y^1x_1P_y^2x_2\cdots P_y^kx_k$ is an HP of $G[P_{xy}]$ where x_i's are vertices in $P_{xy}\cap X$ for $1\le i\le k$ and P_x is an (X)-path visits all vertices $X-\{x_i:1\le i\le k\}$. Then, P_sP' is an HP of G whose endpoints are not in Y. ∎

Though the following lemma is inspired by those given in [20], it is slightly different from the original form in [20] and leads to a more simple algorithm.

Lemma 4 *Let $X'\subset X$ and $G'=G-(X'\cup Y)$.*
(1) *If G has an HC, then $|X|\ge\pi_0(Y)+1$.*
(2) *If $|X|\ge\pi_0(Y)+1$ and $|X'|=\pi_0(Y)$, then G has an HC iff G' has an HC.*

Proof. (1) Suppose C is an HC of G. Let k be The number of (Y)-maximal paths in C. It is easy to see that $k\ge\pi_0(Y)$. To connect these k (Y)-maximal paths into an HC, there are at k $(V-Y)$-maximal paths in C. These $(V-Y)$-maximal paths starts from a vertex in X and ends at a vertex in X. In other words, each of these $(V-Y)$-maximal paths visits at least one vertex in X. Since $Z\neq\emptyset$, at least one of these $(V-Y)$-maximal paths visits a vertex in Z. A $(V-Y)$-maximal path that visits a vertex in Z visits at least two vertices in X. Thus, $|X|>k$.

(2) Suppose G has an HC. It is easy to see that G has an HP, denoted by $\mathcal{P}$, that starts from a vertex $s\in Z$ and ends at a vertex $t\in X$ where s and t are adjacent in G. Obviously, $\mathcal{P}-(X'\cup Y)$ is an HP of G'. Hence G' has an HC.

Conversely, suppse G' has an HC. Then, G' has an HP, denoted by $\mathcal{P}^*$ that starts from a vertex in x and ends at a vertex in Z. By the definition of X and Y, there exists an HP, denoted by $\mathcal{P}'$, of $G[X'\cup Y]$ that starts from a vertex in X and ends at a vertex in Y. Clearly, $\mathcal{P}'\mathcal{P}^*$ is an HP of G. Since X and Z are joint, G has an HC. ∎

Now, we can explain the basic ideas of the algorithms. If G is a cograph, then we can solve the HC problem by the algorithm given in [8]. By Theorem 1, we can find vertex sets X, Y, and Z satisfying the conditions given at the begining of this section. By Lemma 4, we can solve the HC problem for G by solving the HC problem for G' where the number of vertices of G' is less than that of G. By repeatedly applying Lemma 4, eventually G' becomes a cograph. This leads to a polynomial time algorithm for the HC problem in Ptolemaic graphs. The time complexity of this algorithm depends on how fast we can find vertex sets X and Y. Theorem 1 suggests a very efficient implementation for computing vertex sets X and Y by using a hanging of G. A hanging of G can be computed in $O(n+m)$ time. Suppose G has a hanging h_u at vertex u such as $L_0(u)$, $L_1(u)$,..., $L_t(u)$. Then, X will be a subset of $L_{t-1}(u)$ and Y will be a subset of $L_t(u)$. Since we delete vertices in X' and Y to obtain G', we can obtain a hanging for G' from the hanging of G in $O(|X'|+|Y|)$ time where X' is the set of vertices in X that are removed to obtained G'. This leads to a linear time algorithm for the HC problem in Ptolemaic graphs. In the following, we prove lemmas necessary for developing our algorithm for the HP, (HP,s), and (HP,s,t) problems in Ptolemaic graphs by using the same approach.

Lemma 5 *Let $X'\subset X$, $s'\in X-X'$, and $G'=G-(X'\cup Y)$.*

(1) *If G has an HP, then* $|X| \geq \pi_0(Y)$.
(2) *Suppose* $|X| = \pi_0(Y)$ *and* $|X'| = \pi_0(Y) - 1$.
Then, G has an HP iff G' has an (HP, s').
(3) *Suppose* $|X| > \pi_0(Y)$ *and* $|X'| = \pi_0(Y)$.
Then, G has an HP iff G" has an HP.

Proof.

(1) This statement can be proved by arguments similar to those for proving statement (1) of Lemma 4.

(2) Suppose $\mathcal{P}$ is an HP of G. By arguments similar to those for proving statement (1) of Lemma 4, we can prove that $|X| > \pi_0(Y)$ if both endpoints of $\mathcal{P}$ is not in Y. Thus, at least one endpoint of $\mathcal{P}$ is in Y. By statement (1) of Lemma 3, we may assume that $\mathcal{P}$ has one endpoint in Y and the other endpoint not in Y. Withoout loss of generality, assume that $\mathcal{P}$ starts from s and ends at vertex t in Y. By Lemma 2, we may assume that $\mathcal{P}$ is (X, Y)-canonical. Thus, $\mathcal{P} - (X' \cup Y)$ is an HP of G'. Besides, $\mathcal{P} - (X' \cup Y)$ ends at a vertex in X since $|X'| = \pi_0(Y) - 1$. In other words, $\mathcal{P} - (X' \cup Y)$ is an (HP, s') of G'.

Conversely, suppose $\mathcal{P}'$ is an (HP, s') of G' and $\mathcal{P}'$ ends at vertex s'. Since $|X'| = \pi_0(Y) - 1$ and X and Y are joint, there is an HP, denoted by $\mathcal{P}''$, of $G[X' \cup Y]$ with both endpoints in Y. Thus, $\mathcal{P}'\mathcal{P}''$ is an HP of G.

(3) Suppose $\mathcal{P}$ is an HP of G where $\mathcal{P}$ starts from s and ends at t. By Lemma 3 (2), neither s nor t is in Y. By Lemma 2, assume that $\mathcal{P}$ is (X, Y)-canonical. Hence $\mathcal{P} - (X' \cup Y)$ is an HP of G'.

Conversely, suppose $\mathcal{P}'$ is an HP of G'. Since $|X'| = \pi_0(Y)$, there is an HP, denoted by $\mathcal{P}$", of $G[X' \cup Y]$ with one endpoint in X' and the other endpoint in Y. Since $X - X'$ and Z are not empty, there is an edge (x, z) in $\mathcal{P}'$ with $x \in X$ and $z \in Z$. Let $\mathcal{P}' = P_1P_2$ such that P_1 ends at a vertex in Z and P_2 starts from a vertex in X. Then, $P_1\mathcal{P}"P_2$ is an HP of G. ∎

Lemma 6 *Suppose G has an* (HP, s).
(1) *If* $s \in Y$, *then* $|X| \geq \pi_1(Y, s)$.
(2) *If* $s \in X$, *then* $|X| > \pi_0(Y)$.
(3) *If* $s \in V - (X \cup Y)$, *then* $|X| \geq \pi_0(Y)$.

Proof. This statement can be proved by arguments similar to those for proveing statement (1) of Lemma 4. ∎

Lemma 7 *Let* $X' \subset X$, $s' \in X - X'$, *and* $G' = G - (X' \cup Y)$.
(1) *Suppose* $s \in Y$, $|X| \geq \pi_1(Y, s)$ *and* $|X'| = \pi_1(Y, s) - 1$.
Then, G has an (HP, s) iff G' has a (HP, s').
(2) *Suppose* $s \notin (X \cup Y)$, $|X| = \pi_0(Y)$ *and* $|X'| = \pi_0(Y) - 1$.
Then, G has an (HP, s) iff G' has an (HP, s, s').
(3) *Suppose* $s \notin Y$, $|X| \geq \pi_0(Y) + 1$ *and* $|X'| = \pi_0(Y)$.
Then, G has an (HP, s) iff G' has an (HP, s).

Proof. (1) Suppose $\mathcal{P}$ is an (HP, s) of G. By Lemma 3 (3), we may assume that at most one endpoint of $\mathcal{P}$ is in Y. Withoout loss of generality, assume that $\mathcal{P}$ starts from t not in Y and ends at vertex s in Y. By Lemma 2, we may assume that $\mathcal{P}$ is (X, Y)-canonical. Thus, $\mathcal{P} - (X' \cup Y)$ is an HP of G'. Since $|X'| = \pi_1(Y, s) - 1$ and $\mathcal{P}$ has at least $\pi_1(Y, s)$ (Y)-maximal paths, $\mathcal{P} - (X' \cup Y)$ ends at a vertex in X. By the definition of X, Y, and Z, we may assume that $\mathcal{P} - (X' \cup Y)$ ends at vertex s'.

Conversely, suppose $\mathcal{P}'$ is an (HP, s') of G', $\mathcal{P}'$ starts from $s' \in X - X'$, and $|X'| = \pi_1(Y, s) - 1$. Since X is a clique and X and Y are joint, there is an HP, denoted by $\mathcal{P}''$, of $G[X' \cup Y]$ such that $\mathcal{P}''$ starts from s, ends at a vertex in Y. Thus, $\mathcal{P}"\mathcal{P}'$ is an (HP, s) of G since X and Y are joint.

(2) Suppose $\mathcal{P}$ is an (HP, s) of G. By Lemma 2, we may assume that $\mathcal{P}$ is (X, Y)-canonical (HP, s), starts from vertex s and ends at a vertex t. Path $\mathcal{P}$ has at least $\pi_0(Y)$ (Y)-maximal paths. Since $|X| = \pi_0(Y, s)$ and $Z \neq \emptyset$, we have that $t \in Y$. Thus, $\mathcal{P} - (X' \cup Y)$ is an HP of G'. Since $|X'| = \pi_0(Y) - 1$ and $\mathcal{P}$ has at least $\pi_0(Y)$ (Y)-maximal paths, $\mathcal{P} - (X' \cup Y)$ ends at a vertex in X. By the definition of X, Y, and Z, we may assume that $\mathcal{P} - (X' \cup Y)$ ends at vertex s'. In other words, $\mathcal{P} - (X' \cup Y)$ is an (HP, s, s') of G'.

Conversely, by arguments similar to those for proving statement (1), we can show that if G' has an (HP, s, s'), then G has an (HP, s).

(3) Suppose $\mathcal{P}$ is an (HP, s) of G. By Lemma 3 (3) and 2, we may assume that $\mathcal{P}$ is (X, Y)-canonical (HP, s) and neither of its endpoints is in Y. Hence $\mathcal{P} - (X' \cup Y)$ is an (HP, s) of G'.

Conversely, suppose $\mathcal{P}'$ is an (HP, s) of G'. Since $Z \neq \emptyset$, we have that $\mathcal{P}' = P_1P_2$ such that P_1 ends at a vertex in Z and P_2 starts from a vertex in X. Since $|X'| = \pi_0(Y)$, there is an HP, denoted by $\mathcal{P}''$, of $G[X' \cup Y]$ such that $\mathcal{P}$" starts from a vertex in X and ends at a vertex in Y. Thus, $P_1\mathcal{P}"P_2$ is an (HP, s) of G since X and Y are joint and X and Z are joint. ∎

Lemma 8 *Suppose G has an* (HP, s, t).
(1) *If* $\{s, t\} \subseteq Y$, *then* $|X| \geq 2$ *and* $|X| \geq \pi_2(Y, s, t)$.
(2) *If* $s \in Y$ *and* $t \notin Y$, *then* $|X - t| \geq \pi_1(Y, s)$.
(3) *If* $s \in X$ *and* $t \notin Y$, *then* $|X - s - t| \geq \pi_0(Y)$.
(4) *If* $\{s, t\} \cap (X \cup Y) = \emptyset$, *then* $|X| \geq \pi_0(Y) + 1$.

Proof. This statement can be proved by arguments similar to those for proveing statement (1) of Lemma 4. ∎

Lemma 9 *Let $X' \subset X$, $s \notin X'$, $t \notin X'$, $G' = G - (X' \cup Y)$, and s' and t' be any two distinct vertices in $X - X'$.*
(1) *Suppose $\{s,t\} \subseteq Y$, $\pi_2(Y,s,t) = 1$, $|X| \geq 2$.*
Then, G has an (HP,s,t) iff $G[V - Y]$ has an (HP,s',t').
(2) *Suppose $\{s,t\} \subseteq Y$, $\pi_2(Y,s,t) > 1$, $|X| \geq \pi_2(Y,s,t)$ and $|X'| = \pi_2(Y,s,t) - 2$.*
Then, G has an (HP,s,t) iff G' has an (HP,s',t').
(3) *Suppose $t \in Y$, $s \notin Y$, $|X - s| \geq \pi_1(Y,t)$, and $|X'| = \pi_1(Y,t) - 1$.*
Then, G has an (HP,s,t) iff G' has an (HP,s,t').
(4) *Suppose $\{s,t\} \cap Y = \emptyset$, $|X| \geq \pi_0(Y)+1$, $|X-s-t| \geq \pi_0(Y)$ and $|X'| = \pi_0(Y)$.*
Then, G has an (HP,s,t) iff G' has an (HP,s,t).

Proof. (1) Suppose $\mathcal{P}$ is an HP of G. By the definition of X and Y, we may let

$$\mathcal{P} = P_y^1 P_{xx}^1 P_y^2 P_{xx}^2 \cdots P_y^{k-1} P_{xx}^{k-1} P_y^k$$

where each P_y^i is a (Y)-maximal path and each P_{xx}^i does not visits any vertex in Y, starts from a vertex in X and ends at a vertex in X. It is easy to see that $P_{xx}^1 P_{xx}^2 \cdots P_{xx}^{k-1}$ is an HP of G' with both endpoints in X which can be replaced by s' and t'. Thus G' has an (HP,s',t'). Conversely, suppose $\mathcal{P}'$ is an (HP,s',t') of G'. Since $\pi_2(Y,s,t) = 1$, there is an (HP,s,t), $\mathcal{P}''$, of $G[Y]$. Let $\mathcal{P}'' = s\mathcal{P}^*$. Then, $s\mathcal{P}'\mathcal{P}^*$ is an (HP,s,t) of G, since X and Y are joint.

(2) Suppose $\mathcal{P}$ is an HP of G. By Lemma 2, we may assume that $\mathcal{P}$ is an (X,Y)-canonical (HP,s,t). Path $\mathcal{P}$ has at least $\pi_2(Y,s,t)$ (Y)-maximal paths. Since $|X'| = \pi_2(Y,s,t) - 2$, $\mathcal{P} - (X' \cup Y)$ is an HP of G' with both endpoints in $X - X'$ which can be replaced by s' and t'. Thus G' has an (HP,s',t').

Conversely, suppose $\mathcal{P}'$ is an (HP,s',t') of G'. Since $|X'| = \pi_2(Y,s,t) - 2$, we can cover $G[X' \cup Y]$ by two vertex disjoint paths $\mathcal{P}_1$ and $\mathcal{P}_2$ with all their endpoints in Y such that $\mathcal{P}_1$ starts from vertex s, and $\mathcal{P}_2$ ends at vertex t. Thus, $\mathcal{P}_1\mathcal{P}'\mathcal{P}_2$ is an (HP,s,t) of G, since X and Y are joint.

(3) Suppose G has an (HP,s,t). By Lemma 2, we may assume that it is (X,Y)-canonical. Since $|X'| = \pi_1(Y,t) - 1$, path $\mathcal{P} - (X' \cup Y)$ is an HP of G' that starts from vertex s, ends at a vertex in X which can be replaced by vertex t'. In other words, $\mathcal{P} - (X' \cup Y)$ is an (HP,s,t') of G'. Thus G' has an (HP,s,t').

Conversely, suppose $\mathcal{P}'$ is an (HP,s,t') of G' that starts from vertex s and ends at vertex t'. Since $|X'| = \pi_1(Y,t) - 1$, there is an (HP,t) $\mathcal{P}''$ of $G[X' \cup Y]$ with both endpoints in Y. Thus, $\mathcal{P}'\mathcal{P}''$ is an (HP,s,t) of G, since X and Y are joint.

(4) Suppose G has an (HP,s,t). By Lemma 2, we may assume that it is (X,Y)-canonical. Since $\{s,t\} \cap Y = \emptyset$, and $|X'| = \pi_0(Y)$, $\mathcal{P} - (X' \cup Y)$ is an (HP,s,t) of G'.

Conversely, suppose $\mathcal{P}'$ is an (HP,s,t) of G'. Since $|X'| = \pi_0(Y)$, there is an HP, denoted by $\mathcal{P}$", of $G[X' \cup Y]$ that starts from a vertex in X and ends at a vertex in Y. Note that there exist vertices $x \in (X - X')$ and $z \in Z$ such that (x,z) is an edge of $\mathcal{P}'$, since $(X - X')$ and Z are not empty. Let $\mathcal{P}' = P_1P_2$ such that P_1 ends at a vertex in Z and P_2 starts from a vertex in X. Thus, $P_1\mathcal{P}$"P_2 is an (HP,s,t) of G, since X and Y are joint. ∎

In light of previous lemmas and theorem, we have the algorithm shown in Figure 1. for the Hamiltonian path problem on Ptolemaic graphs.

Theorem 10 *Algorithm HP-pt solves the Hamiltonian path problem for Ptolemaic graphs in linear time.*

Proof. In the algorithm shown in Figure 1, we build the hanging h_u by a breadth-first search, and use bucket sort to sort $\mathcal{F}$. Next, since all the three parameters $\pi_0(H)$, $\pi_1(H,s)$ and $\pi_2(H,s,t)$ can be determined in linear time for any cograph H [8, 17, 20]. Thus each level L_i is emptied during the i-th iteration of the "for" loop in $O(|L_i| + |E(G[L_i])|)$ time. So the linearity of the whole algorithm follows. Finally, the correctness of the algorithm follows from Lemma 5 to 9. ∎

With the aid of previous lemmas and corollary, one can easily modify algorithm HP-pt to conclude the following corollary.

Corollary 11 *There exists a linear-time algorithm for determining whether or not a Ptolemaic graph G has an HC (respectively, (HP,s), (HP,s,t) for any vertices s, t in G).*

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Algorithm HP-pt. Determine whether or not a connected Ptolemaic graph has a HP.
Input: A connected Ptolemaic graph $G = (V, E)$.
Output: Determine whether or not G has a Hamiltonian path.
Method.

$C \leftarrow \emptyset$; $p \leftarrow 0$;
determine the hanging $h_u = (L_0, L_1, \ldots, L_f)$ of G at a vertex u;
for $i = f$ **to** 1 **step -1 do**
{ let $\mathcal{F} = \{A_1, A_2, \ldots, A_j\}$ be the components of $G[L_i]$;
sort $\mathcal{F}$ such that $|N_{L_{i-1}}(A_{i_1})| \leq |N_{L_{i-1}}(A_{i_2})| \leq \ldots \leq |N_{L_{i-1}}(A_{i_j})|$;
if $i = 1$ **then** { $j \leftarrow 1$; $A_{i_1} \leftarrow L_1 + u$; }
for $k = 1$ **to** j **do**
{ $Y \leftarrow A_{i_k}$; $X \leftarrow N_{L_{i-1}}(Y)$;
Case 1: $C = \emptyset$
if $i = 1$ **and** $\pi_0(Y) \neq 1$ **then GOTO** (*);
if $|X| < \pi_0(Y)$ **then GOTO** (*) **else** { **if** $|X| = \pi_0(Y)$ **then** $p \leftarrow \pi_0(Y) - 1$ **else** $p \leftarrow \pi_0(Y)$; }
let X' be any subset of X with $|X'| = p$ and $s' \in (X - X')$;
if $p = \pi_0(Y) - 1$ **then** $C \leftarrow \{s'\}$;
Case 2: $|C| = 1$ (say $C = \{s\}$)
if $i = 1$ **then** $\pi_1(Y, s) \neq 1$ **then GOTO** (*);
if $s \in Y$ **then** { **if** $|X| < \pi_1(Y, s)$ **then GOTO** (*) **else** $p \leftarrow \pi_1(Y, s) - 1$; }
if ($s \notin Y$ **and** $|X| < \pi_0(Y)$) **or** ($s \in X$ **and** $|X| = \pi_0(Y)$) **then GOTO** (*);
if $s \notin (X \cup Y)$ **and** $|X| = \pi_0(Y)$ **then** $p \leftarrow \pi_0(Y) - 1$;
if $s \notin Y$ **and** $|X| \geq \pi_0(Y) + 1$ **then** $p \leftarrow \pi_0(Y)$;
let X' be any subset of X with $|X'| = p$ and $t' \in (X - X' - C)$;
if $p = \pi_1(Y, s) - 1$ **then** $C \leftarrow \{t'\}$ **else** { **if** $p = \pi_0(Y) - 1$ **then** $C \leftarrow \{s, t'\}$; }
Case 3: $|C| = 2$ (say $C = \{s, t\}$ and w.l.o.g. say $s \in Y$ as $|C \cap Y| = 1$)
if $i = 1$ and $\pi_2(Y, s, t) \neq 1$ **then GOTO** (*);
if $C \subseteq Y$ **and** ($|X| < 2$ **or** $|X| < \pi_2(Y, s, t)$) **then GOTO** (*);
if $|C \cap Y| = 1$ **and** $|X - s - t| \geq \pi_1(Y, s)$ **then GOTO** (*);
if $|C \cap X| \geq 1$ **and** $C \cap Y = \emptyset$ **and** $|X - s - t| \geq \pi_0(Y)$ **then GOTO** (*);
if $C \cap (X \cup Y) = \emptyset$ **and** $|X| \geq \pi_0(Y) + 1$ **then GOTO** (*);
if $C \subseteq Y$ **and** $\pi_2(Y, s, t) = 1$ **and** $|X| \geq 2$ **then** $p \leftarrow 0$;
if $C \subseteq Y$ **and** $\pi_2(Y, s, t) > 1$ **and** $|X| \geq \pi_2(Y, s, t)$ **then** $p \leftarrow \pi_2(Y, s, t) - 2$;
if $|C \cap Y| = 1$ **and** $|X - t| \geq \pi_1(Y, s)$ **then** $p \leftarrow \pi_1(Y, s) - 1$;
if $C \cap Y = \emptyset$ **and** $|X| \geq \pi_0(Y) + 1$ **and** $|X - C| \geq \pi_0(Y)$ **then** $p \leftarrow \pi_0(Y)$;
let X' be any subset of X with $|X'| = p$ and s', $t' \in (X - X' - C)$;
if $p = 0$ **or** $p = \pi_2(Y, s, t) - 2$ **then** $C \leftarrow \{s', t'\}$ **else** { **if** $p = \pi_1(Y, s) - 1$ **then** $C \leftarrow \{s', t\}$; }
$L_{i-1} \leftarrow L_{i-1} - X'$; }
}
print "G has Hamiltonian path"; **exit**;
(*)**print** "G has no Hamiltonian path";

Figure 1: The algorithm for the Hamiltonian path on Ptolemaic grpahs