Cycle Embedding on the Edge Fault Star Graphs

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Abstract

The star graph is one of the famous interconnection networks. Edge fault tolerance is an important issue for a network. And the cycle embedding problem is widely discussed in many researches. In this paper, we show that the n-dimensional star graph can be embedding cycles of even length from 6 to $n!$ when the number of edge fault does not exceed $n-3$. Since the graph is bipartite and $(n-1)$ regular, our result is optimal.

Keywords: star graph, cycle embedding, fault tolerant, pancyclic.

1 Introduction

Network topology is a crucial factor for a network since it determines the performance of the network. For convenience of discussing their properties, networks are usually represented by graphs. In this paper, a network topology is represented by a simple undirected graph, which is loopless and without multiple edges. For the graph definition and notation we follow [2]. $G = (V, E)$ is a graph if V is a finite set and E is a subset of $\{(a, b) \mid a \neq b \in V\}$, where (a, b) denotes an unordered pair. We call V the *vertex set* and E the *edge set*. We say that vertices a and b are adjacent if and only if $(a, b) \in E$. A path is a sequence of adjacent vertices, denoted by $\langle v_0, v_1, \cdots, v_k \rangle$, in which $v_0, v_1, \cdots, v_k$ are distinct except that possibly $v_0 = v_k$. The length of the path is k. We say that the path is a cycle if $v_0 = v_k$. A path (or a cycle) is hamiltonian with respect to a graph G if it crosses all vertices of G.

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To find a cycle of given length in a graph is a cycle embedding problem. Particularly, to find cycles of all lengthes in a graph, i.e., of length from 3 to the number of vertices in the graph, is the pancyclic problem. In this paper, we focus on such a problem on the star graphs proposed by Akers et. al [1]. Since the star graphs are bipartite graphs, they contain no odd length cycles. Moreover, the minimum length of cycles of the star graphs is 6. Jwo et. al [5] showed that an n-dimensional star graph contains cycles of even length from 6 to $n!$. Thus, we may say that the star graphs are weak even pancyclic.

However, there is no result about such a property on the faulty star graphs. Since components in a network would fail sometimes, to study the graphs with faults in the researches is more practice. In this study, we discuss the even cycles embedding problem on the edge fault star graphs. We show that an n-dimensional star graph contains cycles of even length from 6 to $n!$ when the number of edge faults does not exceed $n-3$. In the next section, we introduce the definition of star graphs. In Section 3, we show the main result in the finally.

2 Definition and Basic Properties

The following is the definition of the star graphs. For convenience, we always use $v_1v_2\cdots v_n$ as the digital representation of vertex v in S_n in this paper.

Definition 1 *The n-dimensional star graph, denoted by S_n, is the graph (V,E), where $V = \{v \mid v \text{ is a permutation of } 1,2,\cdots,n\}$ and $E = \{(u,v) \mid v = u_iu_2\cdots u_{i-1}u_1u_{i+1}\cdots u_n\}$.*

By the definition, S_n contains $n!$ vertices and each vertex is of degree $(n-1)$. For example, 1234 is a vertex in S_4 and connects to 2134, 3214, and 4231. We use $N(u)$ to denote the neighborhood of u, e.g., $N(1234) = \{2134, 3214, 4231\}$. S_1, S_2, and S_3 are a vertex, an edge, and a cycle of length 6, respectively. We show S_4 in Figure 1. It is easy to observe that there are four vertex-disjoint S_3's embedded in S_4. The following lemma states this property.

Lemma 1 *There are n vertex-disjoint S_{n-1}'s embedded in S_n for $n \geq 2$.*

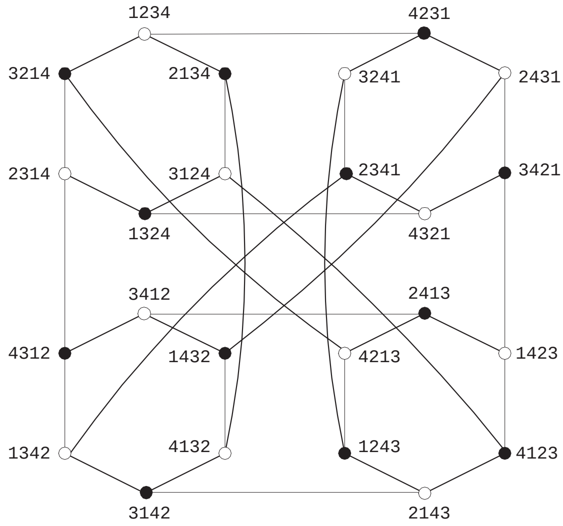

Figure 1: 4-dimensional star graph.

Proof. Let i be some integer between 2 and n. Let $H^{i:j} = (V^{i:j}, E^{i:j})$ for $V^{i:j} = \{u \in V(S_n) \mid u_i = j\}$ and $E^{i:j} = \{(u,v) \in E(S_n) \mid u, v \in V^{i:j}\}$ for $1 \leq j \leq n$. Clearly, $V^{i:1}, V^{i:2}, \cdots, V^{i:n}$ is a partition of $V(S_n)$. It is also not difficult to see that $H^{i:j}$ is isomorphic to S_{n-1}. Thus, the lemma follows. $\square$

We use $S_n^{i:j}$ to denoted the subgraph induced by the vertex set $\{u \mid u_i = j\}$ for $2 \leq i \leq n$. Specifically, we use S_n^j as the abbreviation of $S_n^{n:j}$ and call it as the jth $(n-1)$-dimensional subgraph of S_n.

In our proof, we will use an important property of the edge fault star graphs, called *k-edge-fault hamiltonian laceable.* A graph G is hamiltonian laceable if and only if for any two vertices in G, there is a hamiltonian path of G between them. Furthermore, A graph G is k-edge-fault hamiltonian laceable if and only if for any two vertices in G, there is a hamiltonian path of G between them, where G contains at most k edge faults. The following lemma is proved by [4]

Lemma 2 *S_n is $(n-3)$-edge-fault hamiltonian laceable.*

By the lemma, we have the following result:

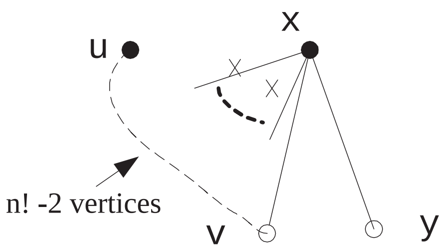

Figure 2: Path of length $n! - 3$ between u and v.

Lemma 3 *Let $n \geq 4$ and $u, v \in V(S_n)$ with u and v in different partite sets. Then there are two paths of length $n! - 3$ and $n! - 1$ between u and v.*

Proof. Since S_n is hamiltonian laceable, there is a path of length $n! - 1$ between u and v. Then consider the path of length $n! - 3$. Let $x \in N(v)$ with $x \neq u$ and $y \in N(x)$ with $y \neq v$. Let $F = \{(x, z) \mid z \in N(x) - \{v, y\}\}$. Then $|F| = n - 3$, u and y are in different partite sets, and x only connects to v and y in $S_n - F$. (See Figure 2.) Since S_n is $(n-3)$-edge fault tolerant hamiltonian laceable, there is a hamiltonian path between u and y in $S_n - F$. Obviously, the vertex sequence $\langle v, x, y \rangle$ is in the path. Thus, the path should be of the form $\langle u, \cdots, v, x, y \rangle$. Then we have a path of length $n! - 3$ in S_n between u and v. $\square$

Note that the result is only applied to the fault-free star graphs. We will use this result in our main proof frequently.

3 Main Result

In this section, we propose our main result. Our proof is by induction. For the base case, we enumerate all required cycles in the following lemma.

Lemma 4 *There is a cycle of each even length from 6 to $n!$ in $S_n - F$, where F is a set of edge faults with $|F| \leq n - 3$ for $n = 3$ and 4.*

Proof. For $n = 3$, S_3 is C_6. Since $|F| = 0$, the lemma follows.

For $n = 4$, $|F| = 1$. Since S_4 is edge-symmetric, we may assume that $F = \{(1243, 3241)\}$. Then we construct cycles of length $6, 8, 10, \cdots, 24$ in the following:

C_6: $\langle 1234, 4231, 2431, 1432, 4132, 2134, 1234\rangle$.

C_8: $\langle 1234, 4231, 2431, 1432, 3412, 2413, 4213, 3214, 1234\rangle$.

C_{10}: Replace $(4231, 2431)$ in C_6 by $\langle 4231, 3241, 2341, 4321, 3421, 2431\rangle$.

C_{12}: Replace $(4231, 2431)$ in C_8 by $\langle 4231, 3241, 2341, 4321, 3421, 2431\rangle$.

C_{14}: Replace $(1432, 4132)$ in C_{10} by $\langle 1432, 3412, 4312, 1342, 3142, 4132\rangle$.

C_{16}: Replace $(1432, 3412)$ in C_{12} by $\langle 1432, 4132, 3142, 1342, 4312, 3412\rangle$.

C_{18}: Replace $(2134, 1234)$ in C_{14} by $\langle 2134, 3124, 1324, 2314, 3214, 1234\rangle$.

C_{20}: Replace $(2413, 4213)$ in C_{16} by $\langle 2413, 1423, 4123, 2143, 1243, 4213\rangle$.

C_{24}: Replace $(3214, 1234)$ in C_{20} by $\langle 3214, 2314, 1324, 3124, 2134, 1234\rangle$.

C_{22}: Replace the subpath $\langle 4231, \cdots, 3412\rangle$ in C_{24} by $\langle 4231, 2431, 3421, 4321, 2341, 1342, 3142, 4132, 1432, 3412\rangle$.

Hence, the lemma follows. □

In fact, the cycles in the above proof are constructed by the method similar to that in Lemma 8 except for C_{22} and we find C_{22} by programs however.

For the inductive step, we need to establish $(n-2)$ disjoint paths crossing the given number of subgraphs to guarantee that there is still at least one of these paths crossing no faulty edge when the number of edge faults in S_n does not exceed $(n-3)$. Moreover, the endpoints of these paths must be in the same subgraph. The following three lemmas discuss how to establish the disjoint paths. Then we use the paths to complete our proof in the latest lemma. For ease of description, we use u_i, u_i^j, and $(u^j)_i^k$ to denote the ith digits of vertices u, u^j, and $(u^j)^k$, respectively. We use P or $\langle v_0, P, v_k\rangle$ to denote the same path where the later points the two endpoints of path P. For two paths $P = \langle x, P, y\rangle$ and $Q = \langle u, Q, v\rangle$ with y and u being adjacent, we use $\langle x, P, y, u, Q, v\rangle$ to denote the path concatenating P with Q.

Lemma 5 *Let x and y be two vertices in S_n for $n \geq 3$ with $x_1 = y_1$. Then $d(x, y) \geq 3$.*

Proof. By definition, $d(x, y) \neq 1$. Suppose that $d(x, y) = 2$. Then there is a vertex z adjacent to x and y. Let $x = z_i z_2 \cdots z_{i-1} z_1 z_{i+1} \cdots z_n$ and $y = z_j z_2 \cdots z_{j-1} z_1 z_{j+1} \cdots z_n$. Since $x \neq y$, $i \neq j$ and thus, $x_1 = z_i \neq z_j = y_1$. We get a contradiction. So $d(x, y) \geq 3$. □

Lemma 6 *Let $x^1, x^2, \cdots x^{n-2} \in V(S_n^i)$ with $x_1^1 = x_1^2 = \cdots = x_1^{n-2}$ for $n \geq 3$. If $u^j \in N(x^j)$ for $1 \leq j \leq n-2$ with $u_1^1 = u_1^2 = \cdots = u_1^{n-2}$, then the $2(n-2)$ vertices are distinct.*

Proof. Consider x^{i_1} and x^{i_2} for $1 \leq i_1 < i_2 \leq n-2$. Since $x_1^{i_1} = x_1^{i_2}$, by Lemma 5, $d(x^{i_1}, x^{i_2}) \geq 3$. It is clearly that $x^{i_1}, x^{i_2}, u_{i_1}$, and u_{i_2} are distinct. □

Lemma 7 *There are $(n-2)$ disjoint paths of length $2m-1$ crossing m $(n-1)$-dimensional subgraphs of S_n such that the endpoints of these paths are in S_n^i for any $1 \leq i \leq n$ and $3 \leq m \leq n$ for $n \geq 3$.*

Proof. Without loss of generality, assume that $i = m$. Let $a^1, a^2, \cdots, a^{n-2} \in V(S_m^n)$ be distinct $(n-2)$ vertices with $a_1^j = 1$ for all $1 \leq j \leq n-2$. Consider the $(n-2)$ paths: $\langle a^j = (a^j)^0, (a^j)^1, (a^j)^2, \cdots, (a^j)^{2m-1}\rangle$ for all $1 \leq j \leq n-2$, where

1. $((a^j)^{2k-2}, (a^j)^{2k-1})$**:** $(a^j)_1^{2k-1} = (a^j)_n^{2k-2}$ and $(a^j)_n^{2k-1} = (a^j)_1^{2k-2}$ for $1 \leq k \leq m$, i.e., exchanging the first digit and the last digit; and

2. $((a^j)^{2k-1}, (a^j)^{2k})$**:** $(a^j)_1^{2k} = (a^j)_l^{2k-1}$ and $(a^j)_l^{2k} = (a^j)_1^{2k-1}$ such that $(a^j)_l^{2k-1} = k+1$ for $1 \leq k \leq m-1$, i.e., exchanging the first digit and the lth digit which is equal to $k+1$. (See example below the proof.)

It is not difficult to check that each path goes though $S_n^m, S_n^1, S_n^2, \cdots, S_n^{m-1}$ and then returns to S_n^m. Clearly, all the paths are of length $2m-1$ and the endpoints of each path are in S_n^m. Since there are exactly two vertices of each path in one $(n-1)$-dimensional subgraph, by Lemma 6, the $2(n-2)$ vertices of all paths in anyone subgraph are distinct. Hence, all the vertices in these $(n-2)$ paths are distinct. $\square$

For example, $n = 6$, $m = 3$, and $a_1 = 124563, a_2 = 125463, a_3 = 142563, a_4 = 145263$. Then we have four disjoint paths as following:

$$\langle 124563, 324561, 234561, 134562, 314562, 214563\rangle,$$

$$\langle 125463, 325461, 235461, 135462, 315462, 215463\rangle,$$

$$\langle 142563, 342561, 243561, 143562, 341562, 241563\rangle,$$

$$\langle 145263, 345261, 245361, 145362, 345162, 245163\rangle.$$

So we may easily find a path of length $2m-1$ crossing m $(n-1)$-dimensional subgraphs in $S_n - F$, where F is a set of edge faults with $|F| \leq n-3$. And such a path uses exactly one edge in each $(n-1)$-dimensional subgraph. Note that the two endpoints of the path are not adjacent except for $m = 3, 4$. Now we show the inductive step:

Lemma 8 *There are cycles of all even length from* 6 *to* $n!$ *in* $S_n - F$*, where* F *is a set of edge faults with* $|F| \leq n-3$ *for* $n \geq 5$.

Proof. Assume that the statement is true for all $3 \leq k \leq n-1$. Since S_n is edge-symmetric, we may assume that there is at least one faulty edge between two $(n-1)$-dimensional subgraphs, i.e., not in any $(n-1)$-dimensional subgraph. Thus, each subgraph contains at most $(n-4)$ faulty edges and is still hamiltonian laceable. Moreover, since $|F| \leq n-3$, there are at least four $(n-1)$-dimensional subgraphs containing no faulty edge. Without loss of generality, assume that S_n^1 and S_n^n contain no faulty edge.

By the hypothesis, we have cycles of each **even length from** 6 **to** $(n-1)!$ in $S_n^1 - F$, i.e., we have cycles of each even length from 6 to $(n-1)!$ in $S_n - F$.

Now we construct cycles of **even length from** $(n-1)!+2$ **to** $(n-1)!+2(n-2)$. Suppose that $l=(n-1)!-4+2m$ for some $3 \le m \le n$. By Lemma 7, there is a path P of length $2m-1$ crossing m $(n-1)$-dimensional subgraphs of $S_n - F$ with two endpoints in S_n^1. Assume that the two endpoints are u and v. By Lemma 3, there is a path Q of length $(n-1)!-3$ in S_n^1 between u and v. Then $\langle u, P, v, Q, u\rangle$ forms a cycle of length $(n-1)!-4+2m$ in $S_n - F$.

Then we construct cycles of **even length from** $(n-1)!+2(n-1)$ **to** $n!$. Let $l = k((n-1)!-2)+2n+2h$ for some $1 \le k \le n-1$ and $0 \le h \le \frac{(n-1)!-2}{2} - 1$ such that $(n-1)!+2(n-1) \le l \le n!$. Then l may be any even integer between $(n-1)!+2(n-1)$ and $n!$. In the following, we construct cycles of length l in $S_n - F$.

By Lemma 7, there is a path $P = \langle u^1, v^2, u^2, v^3, \cdots, v^n, u^n, v^1\rangle$ in $S_n - F$ such that $u^i, v^i \in V(S_n^i)$. Consider two cases:

Case 1: $h \neq 1$**.** If $k \ge 2$, replace each edge (v^i, u^i) in P by a hamiltonian path of $S_n^i - F$ for $2 \le i \le k$. If $h \ge 2$, by the symmetric property of S_{n-1} and hypothesis, we may find a cycle of length $2h+2$ crossing the edge (v^n, u^n). So we may replace the edge (v^n, u^n) in P by a path of length $2h+1$. Now the length of P is $(k-1)((n-1)!-2)+2n-1+2h = l-((n-1)!-1)$. By Lemma 3, there is a path Q of length $(n-1)!-1$ in S_n^1 between v^1 and u^1. Thus, $\langle u^1, P, v^1, Q, u^1\rangle$ forms a cycle of length l in $S_n - F$.

Case 2: $h = 1$**.** If $k \ge 2$, replace each edge (v^i, u^i) in P by a hamiltonian path of $S_n^i - F$ for $2 \le i \le k$. Since there is no fault in S_n^n, there is a cycle of length 6 crossing the edge (v^n, u^n). So we may replace the edge (v^n, u^n) in P by a path of length 5. Now the length of P is $(k-1)((n-1)!-2)+2n-1+2h = l-((n-1)!-3)$. By Lemma 3, there is a path Q of length $(n-1)!-3$ in S_n^1 between v^1 and u^1. Thus, $\langle u^1, P, v^1, Q, u^1\rangle$ forms a cycle of length l in $S_n - F$.

Hence, the lemma follows. $\square$

So we have the following result:

Theorem 1 *There are cycles of all even length from* 6 *to* $n!$ *in* $S_n - F$*, where* F *is a set of edge faults with* $|F| \le n-3$ *for* $n \ge 3$.

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