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A Note on the Inclusion of Human Capital in the Lucas Model

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Abstract

We include human capital in the utility function in an otherwise standard Lucas (1988) model of endogenous growth and show that there may be multiple steady-state equilibria. We also analyze the transitional dynamic properties of this model.

Key words: human capital; endogenous growth; multiple steady-state equilibria; saddle path

JEL classification: C62; J24; O41

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1. Introduction

The model developed by Lucas (1988) is well known in the theoretical literature on human capital accumulation and endogenous growth. In this model, the representative individual allocates labor time between production and learning, and the steady-state equilibrium rate of growth of the economy depends on the allocation of time to acquiring education. This optimal allocation of time in the steady-state equilibrium is unique in the Lucas model. However, this model assumes that individuals derive utility only from consumption. The stock of human capital does not enter as an argument in this utility function.

There is a vast theoretical literature based on the overlapping generation (OLG) framework in which the human capital of offspring is included as an argument in the parental utility function. This literature includes Saint-Paul and Verdier (1993), Eckstein and Zilcha (1994), Selod and Zenou (2003), de la Croix and Doepke (2004), Foster and Rosenzweig (2004), and many others. In none of these models is multiple equilibria obtained as a result of the specific nature of the utility function. However,

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another set of studies dealing with various extensions of the Lucas model do not introduce human capital in the household's utility function. A few Lucas-type models include leisure as an argument in the utility function. The list includes Mino (1999), Benhabib and Perli (1994), Ben-Gad (2003), and de Hek (2005).

In this paper, we introduce human capital as an argument in the utility function of an immortal household such that the marginal utility of consumption varies positively with the stock of human capital. However, we do not consider the individual labor-leisure choice. The individual cares for her own human capital because there are many items that cannot be consumed without literacy or education. She cannot appreciate literary work or artistic creation. She cannot play computer games. Moreover, she may be also ignorant of why and how the consumption of natural resources and the private sector's products create health hazards and environmental pollution. Schultz (1961) noted that the distinguishing feature of investment in human capital compared to investment in physical capital is that investment in human capital not only enhances capabilities but also satisfies preferences. According to Moretti (2005), an increase in the level of education lowers the crime rate and raises political awareness. Gradstein and Justman (2000) points out that human capital helps to build up social capital, enhances social cohesion, and reduces ethnic tension.

The inclusion of human capital in the utility function in the Lucas model may lead to the existence of multiple steady-state equilibria even in the absence of the labor-leisure choice. Kurz (1968) and Liviatan and Samuelson (1969) show the possibility of multiple steady-state equilibria in the one-sector Solow model when the physical capital stock is introduced into the utility function. In some OLG models (e.g., Galor and Zeira, 1993; Banerjee and Newman, 1993; Glomm and Ravikumar, 1995), the possibility of multiple equilibria exists. However, its explanation does not lie in the inclusion of the human capital of offspring in the parental utility function. It is explained by other features, such as credit market imperfection, indivisibilities of investments, and endogenization of public policy.

This paper is organized as follows. In Section 2, we present the basic model and derive the equations of motion. In Section 3, we discuss the properties of steady-state growth equilibrium. In Section 4, we analyze transitional dynamic properties. Concluding remarks are made in Section 5.

2. The Model

In this paper, we consider a standard Lucas model. The dynamic optimization problem of the representative individual in this model is to maximize:

$$\int_0^\infty U(C,H)e^{-\rho t}dt$$

subject to the production function:

$$Y = AK^\alpha (uH)^{1-\alpha} H_A^\gamma ,$$

with A, $\gamma > 0$ and $0 < \alpha < 1$. The dynamic budget constraint is given by:

$$\dot{K} = Y - C,$$

and the human capital accumulation technology is given by:

$$\dot{H} = \delta(1-u)H \quad \text{with} \quad \delta > 0.$$

Here A is the technology level, K is the stock of physical capital, H is the stock of human capital, H_A is the average human capital of all individuals, C is the level of consumption of the representative household, Y is the level of output, u is the fraction of labor time allocated to production, δ is the productivity parameter in the human capital accumulation function, $u(\cdot)$ is the utility function, ρ is the discount rate, α is the capital elasticity of output, and γ is a parameter representing the magnitude of the external effect of human capital. Hence the production function satisfies increasing returns to scale in the presence of the external effect, and if external effect is absent (i.e., $\gamma = 0$) the production function is subject to constant returns to scale.

Introducing human capital, we define the utility function:

$$U(C,H) = \frac{(C^{\mu}H^{1-\mu})^{1-\sigma}}{1-\sigma} \quad \text{with} \quad 0 \le \mu \le 1 \quad \text{and} \quad \sigma > 0.$$

If $\mu = 1$, we obtain the original Lucas model. Here $0 < \mu < 1$ implies that $U_H > 0$. We consider all individuals to be identical so that $H_A = H$. The representative individual solves this optimization problem with respect to the control variables C and u, where K and H are two state variables. However, the individual cannot internalize the externality. The current-value Hamiltonian function is:

$$Z = \frac{(C^{\mu}H^{1-\mu})^{1-\sigma}}{1-\sigma} + \lambda_K\left[AK^{\alpha}(uH)^{1-\alpha}H_A^{\gamma} - C\right] + \lambda_H\left[\delta(1-u)H\right],$$

where λ_K and λ_H are the co-state variables of K and H, representing the shadow prices of physical and human capital, respectively.

The first-order optimality conditions are given by:

$$\frac{\partial Z}{\partial C} = (C^{\mu}H^{1-\mu})^{-\sigma}\mu C^{\mu-1}H^{1-\mu} - \lambda_K = 0 \tag{1}$$

and

$$\frac{\partial Z}{\partial u} = \lambda_K AK^{\alpha}(1-\alpha)u^{-\alpha}H^{1-\alpha}H_A^{\gamma} - \lambda_H\delta H = 0. \tag{2}$$

Time derivatives of the co-state variables satisfying the optimum growth path are:

$$\dot{\lambda}_K = \rho\lambda_K - \lambda_K \alpha A K^{\alpha-1}(uH)^{1-\alpha} H_A^{\gamma} \tag{3}$$

and

$$\dot{\lambda}_H = \rho\lambda_H - (C^{\mu}H^{1-\mu})^{-\sigma}(1-\mu)C^{\mu}H^{-\mu} - \lambda_K(1-\alpha)AK^{\alpha}u^{1-\alpha}H^{-\alpha}H_A^{\gamma} - \lambda_H\delta(1-u)\,. \tag{4}$$

The transversality condition is:

$$\lim_{t\to\infty} e^{-\rho t}\lambda_K(t)K(t) = \lim_{t\to\infty} e^{-\rho t}\lambda_H(t)H(t) = 0\,.$$

Using equations (1), (2), (3), and (4) and the fact that in equilibrium $H_A = H$, we derive the following equations of motion:

$$\dot{q} = q\left[Au^{1-\alpha}z\left\{\frac{\alpha}{1-\mu(1-\sigma)} - 1\right\} + q + \frac{(1-\mu)(1-\sigma)}{1-\mu(1-\sigma)}\delta(1-\mu) - \frac{\rho}{\{1-\mu(1-\sigma)\}}\right], \tag{5}$$

$$\dot{u} = u\left[\frac{(1-\mu)\delta u^{\alpha}q}{\mu A(1-\alpha)z\alpha} + \frac{\delta}{\alpha} - \frac{(\alpha-\gamma)}{\alpha}\delta(1-u) - q\right], \tag{6}$$

and

$$\dot{z} = z\left[(\alpha-1)Au^{1-\alpha}z + (1-\alpha)q + (1-\alpha+\gamma)\delta(1-u)\right]. \tag{7}$$

Here q and z are two ratio variables given by:

$$q = C/K \quad \text{and} \quad z = K^{\alpha-1}H^{1-\alpha+\gamma}\,.$$

Note these reduce to the equations of motion in Benhabib and Perli (1994) when $\mu = 1$.

3. Steady-State Equilibrium

In the steady-state equilibrium, $\dot{q} = \dot{u} = \dot{z} = 0$. We denote the steady-state equilibrium values of q, u, and z by q^*, u^*, and z^*, respectively. Using equations (5), (6), and (7) we obtain a description of the steady-state:

$$q^* = \frac{\rho}{\alpha} - \frac{\delta(1-u^*)}{\alpha}\left[(\alpha-\gamma-\sigma) + (1-\sigma)\frac{\mu\gamma}{1-\alpha}\right] = \frac{\rho}{\alpha} - A_1\delta(1-u^*)\,, \tag{8}$$

where

$$A_1 = \frac{1}{\alpha}\left[(\alpha-\gamma-\sigma) + (1-\sigma)\frac{\mu\gamma}{1-\alpha}\right],$$

$$z^* = \frac{1}{Au^{*1-\alpha}}\left[\frac{\rho}{\alpha} + \left\{\frac{1-\alpha+\gamma}{1-\alpha} - A_1\right\}\delta(1-u^*)\right], \tag{9}$$

and

$$au^{*2} + bu^* + c = 0\,, \tag{10}$$

where

$$a = \delta^2\left[\frac{(1-\mu)A_1}{\mu(1-\alpha)\alpha} - \left\{\left(\frac{\alpha-\gamma}{\alpha}\right) - A_1\right\}\left\{\frac{1-\alpha+\gamma}{1-\alpha} - A_1\right\}\right],$$

$$b = \frac{(1-\mu)\delta}{\mu(1-\alpha)\alpha}\left\{\frac{\rho}{\alpha} - A_1\delta\right\} - \left(\frac{\delta-\rho}{\alpha}\right)\delta\left\{\frac{1-\alpha+\gamma}{1-\alpha} - A_1\right\}$$
$$+\left\{\frac{\alpha-\gamma}{\alpha} - A_1\right\}\delta\left\{\frac{\rho}{\alpha} + \left\{\frac{1-\alpha+\gamma}{1-\alpha} - A_1\right\}2\delta\right\},$$

and

$$c = \left\{\frac{\rho}{\alpha} + \left\{\frac{1-\alpha+\gamma}{1-\alpha} - A_1\right\}\delta\right\}\left\{\frac{\delta-\rho}{\alpha} - \left\{\frac{\alpha-\gamma}{\alpha} - A_1\right\}\delta\right\}.$$

Therefore, q^* and z^* are uniquely related to u^* by equations (8) and (9). If there is a unique value of u^*, then q^* and z^* are also unique. From equation (6), we have in steady state:

$$\frac{(1-\mu)\delta q^*}{\mu A u^{*-\alpha}(1-\alpha)\alpha z^*} = \frac{\alpha-\gamma}{\alpha}\delta(1-u^*) + q^* - \frac{\delta}{\alpha}.$$

Substituting the steady-state equilibrium value of q from equation (8) in this equation we have:

$$\frac{(1-\mu)\delta q^*}{\mu A u^{*-\alpha}(1-\alpha)\alpha z^*} = \frac{\delta(1-u^*)}{\alpha}\left[\sigma - (1-\sigma)\frac{\mu\gamma}{1-\alpha}\right] + \frac{\rho-\delta}{\alpha}. \tag{11}$$

Since the left-hand side is positive, the right-hand side must also be positive. Hence:

$$\delta(1-u^*) > \frac{\delta-\rho}{\sigma - (1-\sigma)\dfrac{\mu\gamma}{1-\alpha}}.$$

If $\rho < \delta$ and $\sigma \geq 1$, the right-hand side is positive and we can write:

$$\delta(1-u^*) > \frac{\delta-\rho}{\sigma-(1-\sigma)\frac{\mu\gamma}{1-\alpha}} > \frac{\delta-\rho}{\sigma-(1-\sigma)\frac{\gamma}{1-\alpha}} . \tag{12}$$

The term on the extreme left of (12) is the rate of growth of human capital in our model, while the term on the extreme right is the corresponding growth rate in the Lucas model. If $\rho > \delta$ and $\sigma \geq 1$, the last two terms are negative. However, the term in the extreme left is positive if $0 < u^* < 1$, and this may hold because the left-hand side of (11) is positive for $0 < \mu < 1$. So even in that case:

$$\delta(1-u^*) > \frac{\delta-\rho}{\sigma-(1-\sigma)\frac{\gamma}{1-\alpha}} .$$

This is not possible in the Lucas because $\mu = 1$ there and hence:

$$\delta(1-u^*) = \frac{\delta-\rho}{\sigma-(1-\sigma)\frac{\gamma}{1-\alpha}} .$$

This implies that $\delta > \rho$ is necessary for u^* to satisfy $0 < u^* < 1$ in the Lucas model. Consequently, in our model, the rate of growth of human capital is higher than that in the Lucas model for all interior solutions of u^*. We obtain the following proposition.

***Proposition 1*:** The steady-state equilibrium rate of growth of human capital in our model is greater than that in the Lucas model for an interior solution of u^*. It may be positive even for $\rho > \delta$, while in the Lucas model it is positive only if $\rho < \delta$.

The intuition behind this result is very simple. In our model, human capital enters not only into the production function as a productive input but also into the utility function as an argument with positive marginal utility. In the Lucas model, the marginal utility of human capital is always zero. Therefore, the household allocates more labor time to human capital accumulation in the present model than in the Lucas model.

Also note that Proposition 1 is valid for values of u^* satisfying $0 \leq u^* \leq 1$. We have assumed the existence of an interior solution because Lucas did the same. However, if $u^* = 1$, then $\dot{H} = 0$ and the rate of growth in the steady-state equilibrium is zero in both this model and in the Lucas model. If $u^* = 0$, the two models are identical to the one-sector Solow model with no human capital accumulation.

Two positive solutions of u^* emerge from equation (10) if (i) $a > 0$, $c > 0$, and $b < 0$ or (ii) $a < 0$, $c < 0$, and $b > 0$. The condition for the larger positive root to be less than 1 is $a + b + c \geq 0$ in case (i) and is $a + b + c < 0$ in case (ii). In these two situations, both the roots lie between 0 and 1; see Appendix A.1. Here,

$$a+b+c=\frac{\rho}{\alpha^2}\left[\frac{\delta(1-\mu)}{\mu(1-\alpha)}+(\delta-\rho)\right].$$

When $\sigma \geq 1$, a and c are negative and b is positive; see Appendix A.2. In this case, both solutions of u^* are positive. The condition that both the roots lie between 0 and 1 is $a+b+c \leq 0$, and this is satisfied if:

$$\rho \geq \delta\left[\frac{1-\mu}{\mu(1-\alpha)}+1\right].$$

***Proposition 2*:** If $\sigma \geq 1$ and $\rho \geq \delta\{[(1-\mu)/\mu(1-\alpha)]+1\}$, there exist two solutions of u^* satisfying $0 < u^* < 1$.

In the Lucas model $\mu = 1$, and in this case there is a unique solution of u^*. This is clearly understood by looking at equation (11). Note that when $\sigma \geq 1$ and $\rho \geq \delta\{[(1-\mu)/\mu(1-\alpha)]+1\}$, the growth rate of human capital in the steady-state equilibrium in the Lucas model is negative. But in this model, where the household derives utility from human capital, the rate of growth of human capital is positive in the steady-state equilibrium even in this special case. It is the high positive marginal utility of human capital that induces the individual to allocate a positive fraction of labor time to human capital accumulation even when ρ is high and δ is low. Also, we obtain multiple positive solutions of u^*. When $\sigma \geq 1$ and $\rho < \delta\{[(1-\mu)/\mu(1-\alpha)]+1\}$, then $a < 0$, $c < 0$, $b > 0$, and $a+b+c > 0$. In this case one root lies between 0 and 1 and the other root is greater than 1. Since $u^* > 1$ is not feasible, the solution is unique.

We now try to provide an intuition for the existence of multiple equilibria. The inter-temporal equilibrium point of a differential equation is unique if the derivative is a monotonic function of the dependent variable. In the Lucas model, since human capital does not enter as an argument in the utility function, the relative rate of change in the shadow price of human capital, $\dot{\lambda}_H/\lambda_H$, is independent of the marginal rate of indifferent substitution between consumption and human capital in that model; it is determined only by the marginal productivity of human capital. Once human capital is efficiently allocated between the production sector and the education sector, the rate of change in the shadow price of human capital becomes a constant, and hence $\dot{u}/u$ becomes a monotonic function of u. However, in the present model, the marginal rate of indifferent substitution between consumption and human capital appears to be an important determinant of $\dot{\lambda}_H/\lambda_H$; thus it affects the time path of the human capital allocation variable u. This disturbs the monotonic relationship between $\dot{u}/u$ and u, and the possibility of multiple steady-state equilibria arises. Even, in a one-sector Solow model, multiple steady states occur when the physical capital stock is introduced into the utility function. Kurz (1968) and Liviatan and Samuelson (1972) also obtain this result and call it the "wealth effects" of a stock variable. Our paper basically deals with the wealth effect of the human capital stock.

4. Transitional Dynamics

We now analyze the transitional dynamic properties around the steady-state equilibrium point(s). We consider the system described by equations (5), (6), and (7). Note that this is a system of three differential equations. An initial value of the variable z is historically given and of other two variables q and u can be chosen by the individual. Thus, in order to get the unique saddle path that converges to the steady-state equilibrium point, we need two latent roots of the Jacobian matrix to be positive and one to be negative.

The Jacobian matrix corresponding to the system of differential equations (5), (6), and (7) is:

$$J = \begin{bmatrix} \frac{\partial \dot{q}}{\partial q} & \frac{\partial \dot{q}}{\partial u} & \frac{\partial \dot{q}}{\partial z} \\ \frac{\partial \dot{u}}{\partial q} & \frac{\partial \dot{u}}{\partial u} & \frac{\partial \dot{u}}{\partial z} \\ \frac{\partial \dot{z}}{\partial q} & \frac{\partial \dot{z}}{\partial u} & \frac{\partial \dot{z}}{\partial z} \end{bmatrix},$$

where the elements of the Jacobian in the steady-state equilibrium are given in Appendix A.3. The characteristic equation of the J matrix is given by:

$$\left| J - \lambda I_3 \right| = 0 ,$$

where λ is an eigenvalue of the Jacobian evaluated at steady state. The three characteristic roots can be solved from the equation:

$$a_0 \lambda^3 + b_0 \lambda^2 + a_1 \lambda + b_1 = 0 ,$$

where

$$a_0 = -1 ,$$
$$b_0 = \mathrm{Tr}\ J ,$$
$$a_1 = \frac{\partial \dot{z}}{\partial x}\frac{\partial \dot{x}}{\partial z} + \frac{\partial \dot{x}}{\partial a}\frac{\partial \dot{a}}{\partial x} + \frac{\partial \dot{a}}{\partial z}\frac{\partial \dot{z}}{\partial a} - \frac{\partial \dot{x}}{\partial x}\frac{\partial \dot{a}}{\partial a} - \frac{\partial \dot{z}}{\partial z}\frac{\partial \dot{a}}{\partial a} - \frac{\partial \dot{z}}{\partial z}\frac{\partial \dot{x}}{\partial x} ,$$

and

$$b_1 = \mathrm{Det}\ J .$$

The trace of J is:

$$b_0 = 2\left[\rho - \delta(1-u^*)(1-\sigma)\left(1 + \frac{\mu\gamma}{1-\alpha}\right)\right] - \frac{\delta\gamma}{\alpha} u^* . \qquad (13)$$

Using equation (11) we find that when $0 < \mu < 1$:

$$\rho + \delta(1-u^*)\left(\sigma - (1-\sigma)\frac{\mu\gamma}{1-\alpha}\right) - \delta > 0$$
$$\Rightarrow \rho - \delta(1-u^*)(1-\sigma)\left(1 + \frac{\mu\gamma}{1-\alpha}\right) - \delta u^* > 0 .$$

Equation (13) shows that $b_0 > 0$ when $2\alpha > \gamma$. This means that the trace of J is positive when the external effect is very weak.

The determinant of J is:

$$b_1 = J_{uu}(\alpha - 1)z^* A u^{*1-\alpha} q^* \alpha + J_{uz} z^* q^* \left[A(1-\alpha)^2 u^{*-\alpha} z^* \frac{\alpha}{1-\mu(1-\sigma)} + \delta\left\{(1-\alpha+\gamma) - \frac{(1-\mu)(1-\sigma)(1-\alpha)}{1-\mu(1-\sigma)}\right\}\right]$$
$$- J_{uq} A u^{*1-\alpha} z^* q^* \delta\left[\frac{(1-\alpha)(1-\mu)(1-\sigma)}{1-\mu(1-\sigma)} - (1-\alpha+\gamma)\left(1 - \frac{\alpha}{1-\mu(1-\sigma)}\right)\right].$$

It can be shown that under the sufficient conditions $\sigma \geq 1$ and $\alpha > \gamma$, b_1 is negative; see Appendix A.4. Note that the conditions ensuring $b_0 > 0$ and $b_1 < 0$ are independent of the values of u^* , q^* , and z^* . Therefore, they apply to each of the two steady-state equilibria. We summarize with the following proposition.

***Proposition 3*:** There exists unique equilibrium growth paths converging to each of the two steady-state equilibrium points if $\alpha > \gamma$ and $\sigma \geq 1$.

For each of the two steady-state equilibria, the transitional growth path is unique when the external effect of human capital on production is very weak. This result is similar to that obtained by Benhabib and Perli (1994), Xie (1994), and others, though they analyzed the Lucas model with a unique steady-state equilibrium. Unfortunately, it is very difficult to derive a meaningful condition for the indeterminacy of the transitional growth path when $0 < \mu < 1$. We cannot rule out the possibility of indeterminacy when γ takes a large value. Here $z(0)$ is historically given, while $q(0)$ and $u(0)$ are chosen. Depending on the choice of $q(0)$ and $u(0)$, the initial state trajectory will meet one of the two saddle paths of the two equilibrium points and will converge to the corresponding equilibrium point. Unfortunately, we cannot use a phase diagram to explain the transitional dynamics because it is a 3×3 system.

5. Conclusion

This paper introduces human capital as an argument in the utility function of the household in an otherwise standard Lucas model. It shows that multiple steady-state growth equilibria may exist when the discount rate is very large and/or when the productivity parameter in the human capital accumulation function takes a very small value. So our paper strengthens the importance of the wealth effect of a stock variable in generating multiple equilibria, as shown by Kurz (1968). In a less developed country, the mortality rate is higher than that in a developed economy. Thus, the discount rate is also higher. The human capital accumulation technology is more efficient in an economically advanced economy than in a less developed economy due to differences in educational infrastructure; hence the productivity coefficient of human capital accumulation technology takes a very small value in the latter. Further, the possibility of multiple steady-state growth equilibria appears to be stronger in a less developed economy. The comparison between developed and less developed countries deserves a larger and deeper exposition. What may happen to the long run growth rate if the discount rate declines over time and the productivity coefficient in the human capital accumulation technology increases over time? Will the multiple equilibria persist or will a unique equilibrium appear? These interesting problems may be taken up by future research.

Appendix A.1

When $a < 0$, $c < 0$, and $b > 0$, the larger of the positive roots is:

$$\frac{-b-\sqrt{b^2-4ac}}{2a}.$$

This root is less than 1 if:

$$\frac{-\sqrt{b^2-4ac}}{2a} < 1 + \frac{b}{2a}.$$

Multiplying both sides by $-2a$ and squaring, we obtain:

$$4a(a+b+c) > 0.$$

Since $a < 0$, we must have $a+b+c < 0$. When $a+b+c > 0$, the larger root is greater than 1 and the smaller root is less than 1. Similarly we can show that when $a > 0$, $c > 0$, and $b < 0$, the larger of the positive roots is less than 1 when $a+b+c > 0$.

Appendix A.2

From the expression for A_1, we have:

$$\frac{\alpha-\gamma}{\alpha} - A_1 = \frac{1}{\alpha(1-\alpha)}\left[\sigma(1-\alpha)-(1-\sigma)\mu\gamma\right].$$

This is positive for $\sigma \geq 1$. Also note that for $\sigma > 0$:

$$\frac{1-\alpha+\gamma}{1-\alpha} - A_1 = \frac{1}{\alpha(1-\alpha)}\left[\gamma + \sigma(1-\alpha) - (1-\sigma)\mu\gamma\right]$$
$$= \frac{1}{\alpha(1-\alpha)}\left[\gamma(1-\mu) + \sigma(1-\alpha+\mu\gamma)\right] > 0.$$

Now using the expression for b and this re-expression we have:

$$b = \frac{(1-\mu)\delta}{\mu(1-\alpha)\alpha}\left\{\frac{\rho}{\alpha} - A_1\delta\right\} + \left\{\frac{\alpha-\gamma}{\alpha} - A_1\right\}\delta\frac{\rho}{\alpha} + \left\{\frac{1-\alpha+\gamma}{1-\alpha} - A_1\right\}$$
$$\times\delta\left[2\delta\left\{\frac{\alpha-\gamma}{\alpha} - A_1\right\} - \frac{\delta-\rho}{\alpha}\right].$$

When $\sigma \geq 1$, the first two terms of this expression are negative. The sign of the third term depends on the sign of:

$$2\delta\left\{\left(\frac{\alpha-\gamma}{\alpha}\right) - A_1\right\} - \left(\frac{\delta-\rho}{\alpha}\right),$$

which can be simplified as follows:

$$\frac{\rho}{\alpha} + \frac{\delta}{\alpha}\left[(2\sigma-1) - 2(1-\sigma)\frac{\mu\gamma}{1-\alpha}\right].$$

If $\sigma \geq 1$, this term is negative. Hence $\sigma \geq 1$ is the sufficient condition for b to be positive.

Appendix A.3

The elements of the Jacobian are:

$$J_{qq} = \frac{\partial\dot{q}}{\partial q} = q^*,$$
$$J_{qu} = \frac{\partial\dot{q}}{\partial u} = q^*\left[A(1-\alpha)u^{*-\alpha}z^*\left\{\frac{\alpha}{1-\mu(1-\sigma)} - 1\right\} - \frac{\delta(1-\mu)(1-\sigma)}{1-\mu(1-\sigma)}\right],$$
$$J_{qz} = \frac{\partial\dot{q}}{\partial z} = q^*Au^{*1-\alpha}\left\{\frac{\alpha}{1-\mu(1-\sigma)} - 1\right\},$$
$$J_{uq} = \frac{\partial\dot{u}}{\partial q} = u^*\left[\frac{(1-\mu)\delta u^{*\alpha}}{\mu Az^*(1-\alpha)\alpha} - 1\right],$$

$$J_{uu}=\frac{\partial \dot{u}}{\partial u}=u^*\left[\frac{(1-\mu)\delta u^{*\alpha-1}}{\mu A z^*(1-\alpha)}q^*+\delta\frac{(\alpha-\gamma)}{\alpha}\right],$$

$$J_{uz}=\frac{\partial \dot{u}}{\partial z}=u^*\left[-\frac{(1-\mu)\delta u^{*\alpha}}{\mu A(1-\alpha)\alpha z^{*2}}q^*\right],$$

$$J_{zu}=\frac{\partial \dot{z}}{\partial u}=z^*\left[-(1-\alpha)^2 Au^{*-\alpha}z^*-\delta(1-\alpha+\gamma)\right],$$

$$J_{zz}=\frac{\partial \dot{z}}{\partial z}=A(\alpha-1)u^{*1-\alpha}z^*,$$

and

$$J_{zq}=\frac{\partial \dot{z}}{\partial q}=(1-\alpha)z\ .$$

Using equations (8) and (11) we obtain:

$$J_{uq}=\frac{\delta}{\alpha q^*}\left[(\alpha-\gamma)(1-u^*)-1\right]<0\ .$$

Appendix A.4

The determinant of the Jacobian is:

$$b_1=J_{uu}(\alpha-1)z^*Au^{*1-\alpha}q^*\alpha+J_{uz}z^*q^*\left[A(1-\alpha)^2u^{*-\alpha}z^*\frac{\alpha}{1-\mu(1-\sigma)}\right.$$
$$\left.+\delta\left\{(1-\alpha+\gamma)-\frac{(1-\mu)(1-\sigma)(1-\alpha)}{1-\mu(1-\sigma)}\right\}\right]$$
$$-J_{uq}Au^{*1-\alpha}z^*q^*\delta\left[\frac{(1-\alpha)(1-\mu)(1-\sigma)}{1-\mu(1-\sigma)}-(1-\alpha+\gamma)\left(1-\frac{\alpha}{1-\mu(1-\sigma)}\right)\right].$$

The first term is negative because $J_{uu}>0$ at steady state if $\alpha>\gamma$. The second term is negative because $1-\mu(1-\sigma)>0$, because:

$$(1-\alpha+\gamma)-\frac{(1-\mu)(1-\sigma)(1-\alpha)}{1-\mu(1-\sigma)}=\frac{(1-\alpha)\sigma}{1-\mu(1-\sigma)}+\gamma>0,$$

and because:

$$J_{uz}=\frac{\partial \dot{u}}{\partial z}=-\frac{(1-\mu)\delta u^{*\alpha}u^*q^*}{\mu A(1-\alpha)\alpha z^{*2}}<0\ .$$

Also it can be shown that:

$$\frac{(1-\alpha)(1-\mu)(1-\sigma)}{1-\mu(1-\sigma)} - (1-\alpha+\gamma)\left(1-\frac{\alpha}{1-\mu(1-\sigma)}\right)$$
$$= \frac{1}{1-\mu(1-\sigma)}\left[(1-\alpha)(\alpha-\sigma-\gamma)+\gamma\mu(1-\sigma)\right].$$

This last term is negative if $\sigma \geq 1$. Hence, since the third term of the determinant is also negative, the determinant is negative if $\alpha > \gamma$ and $\sigma \geq 1$.

References

Banerjee, A. V. and A. F. Newman, (1993), "Occupational Choice and the Process of Development," *Journal of Political Economy*, 101(2), 274-298.

Ben-Gad, M., (2003), "Fiscal Policy and Indeterminacy in Models of Endogenous Growth," *Journal of Economic Theory*, 108, 322-344.

Benhabib, J. and R. Perli, (1994), "Uniqueness and Indeterminacy: On the Dynamics of Endogenous Growth," *Journal of Economic Theory*, 63, 113-142.

de Hek, P. A., (2006), "On Taxation in a Two-Sector Endogenous Growth Model with Endogenous Labor Supply," *Journal of Economic Dynamics and Control*, 30(4), 655-685.

Mino, K., (1999), "Non-Separable Utility Function and Indeterminacy of Equilibrium in a Model with Human Capital," *Economics Letters*, 62, 311-317.

de la Croix, D. and M. Doepke, (2004), "Public versus Private Education when Differential Fertility Matters," *Journal of Development Economics*, 73, 607-629.

Eckstein, Z. and I. Zilcha, (1994), "The Effects of Compulsory Schooling on Growth, Income Distribution and Welfare," *Journal of Public Economics*, 54, 339-359.

Foster, A. D. and M. R. Rosenzweig, (2004), "Technological Change and the Distribution of Schooling: Evidence from Green-Revolution India," *Journal of Development Economics*, 74, 87-111.

Galor, O. and J. Zeira, (1993), "Income Distribution and Macroeconomics," *Review of Economic Studies*, 60(1), 35-52.

Glomm, G. and B. Ravikumar, (1992), "Public versus Private Investment in Human Capital: Endogenous Growth and Income Inequality," *Journal of Political Economy*, 100(4), 818-834.

Glomm, G. and B. Ravikumar, (1996), "Endogenous Public Policy and Multiple Equilibria," *European Journal of Political Economy*, 11(4), 653-662.

Gradstein, M. and M. Justman, (2000), "Human Capital, Social Capital, and Public Schooling," *European Economic Review* (Papers and Proceedings), 44, 879-890.

Kurz, M., (1968), "Optimal Economic Growth and Wealth Effects," *International Economic Review*, 9, 348-357.

Liviatan, N. and P. A. Samuelson, (1969), "Notes on Turnpikes: Stable and

Unstable," *Journal of Economic Theory*, 1, 454-475.

Lucas, R. E., (1988), "On the Mechanics of Economic Development," *Journal of Monetary Economics*, 22, 3-42.

Moretti, E., (2003), "Human Capital Externalities in Cities," *NBER Working Paper*, No. 9641.

Saint-Paul, G. and T. Verdier, (1993), "Education, Democracy and Growth," *Journal of Development Economics*, 42, 399-407.

Schultz, T. W., (1961), "Investment in Human Capital," *American Economic Review*, 51, 1-17.

Selod, H. and Y. Zenou, (2003), "Private versus Public Schools in Post-Apartheid South African Cities: Theory and Policy Implications," *Journal of Development Economics*, 71, 351-394.

Xie, D., (1994), "Divergence in Economic Performance: Transitional Dynamics with Multiple Equilibria," *Journal of Economic Theory*, 63, 97-112.