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Testing for Structural Changes in the Presence of Long Memory

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Abstract

We derive the limiting null distributions of the standard and OLS-based CUSUM- tests for a structural change of the coefficients of a linear regression model in the context of long-memory disturbances. We show that both tests behave fundamentally different in a long-memory environment, as compared to short memory, and that long memory is easily mistaken for structural change when standard critical values are employed.

Key words: long memory; structural change; CUSUM- tests

JEL classification: C12; C22; C52

1. Introduction

It is by now well known that long memory and structural change are easily confused [Lobato and Savin (1998), Engle and Smith (1999), Granger and Hyung (1999), Diebold and Inoue (2001), Davidson and Sibbertsen (2002), and many others]. It is therefore of interest to know about the stochastic properties of procedures for detecting and measuring long memory when there is only structural change and for detecting and measuring structural change when there is only long memory.

While the former problem has attracted considerable attention, there has been rather little work on the latter. Nunes et al. (1995), Kuan and Hsu (1998), and Hsu (2001) show that conventional procedures for detecting and dating structural changes tend to find spurious breaks, usually in the middle of the sample, when there is in fact only long memory [Bai 1998)]. As concerns testing, Hidalgo and Robinson (1996) propose a test for structural change which is robust to long memory but assumes a single change point at a known point in time, while Wright (1998) considers the CUSUM-test where no such assumptions are made. Otherwise the literature on testing for structural change in the presence of long memory is rather thin.

Below we follow White (1998) and consider the behaviour of the standard and the OLS-based CUSUM-tests, whose limiting distributions are well understood in the context of various regressor-sequences and i.i.d. or short memory disturbances

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[Krämer et al. (1988) and Ploberger and Krämer (1992, 1996)]. As shown by Wright (1998) for the OLS-based CUSUM-test and the special case of polynomial regressors, these limiting distributions are not robust to departures from short memory—in fact, the OLS-based CUSUM-test has an asymptotic size of unity. This is so because the cumulated sums of the OLS-residuals, although still converging to some well-defined limit process, need a larger normalizing factor for doing so.

The present paper allows for more general regressor sequences and covers the conventional CUSUM-test based on recursive residuals as well. The intuition behind looking at the conventional CUSUM-test was that recursive residuals, other than OLS-residuals, might not inherit the long-memory characteristics of the true disturbances which lead to the exaggerated size of the OLS-based CUSUM-test. However, this is not the case. As is shown below, the null distribution of both the standard and the OLS-based CUSUM-tests tends to infinity, for the same reason already found by Wright (1998): cumulated sums of both types of residuals tend to infinity much faster than in the case of short memory. This is a rather negative result which confirms related theorems from the structural-change-vs-long-memory-literature: Similar to structural change being mistaken for long memory, long memory is likewise easily mistaken for structural change [see also Krämer et al. (2001)]. While there do exist solutions for some special cases, such as Künsch's (1986) procedure to discriminate between long memory and monotonic trends, or Teverovsky and Taqqu's (1997) suggestion based on empirical variances, a general treatment of this problem is still missing and it remains an open problem to efficiently discriminate between the two.

2. Two Unpleasant Theorems

We consider the standard linear regression model

$$y_t = \beta' x_t + \varepsilon_t, \qquad t = 1, 2, \dots, T \tag{1}$$

with non-stochastic, fixed regressors x_t and stationary, zero-mean Gaussian disturbances ε_t. We assume that

$$\frac{1}{T}\sum_{t=1}^{T} x_t \to c < \infty \quad \text{and} \tag{2}$$

$$\frac{1}{T}\sum_{t=1}^{T} x_t x_t' \to Q(\text{finite, nonsingular}). \tag{3}$$

These are standard assumptions in linear regression large sample asymptotics; they exclude trending data, which require separate treatment and proofs which differ from the ones below.

We are concerned with testing the model (1) against the alternative of unspecified structural change in the regression coefficients β. We consider first the OLS-based CUSUM-test as proposed by Ploberger and Krämer (1992). This test rejects the null hypothesis of no structural change for large values of

$$\text{TS} := \sup_{0<\lambda<1} |C_T(\lambda)|, \text{ where} \tag{4}$$

$$C_T(\lambda) := T^{-\frac{1}{2}}\hat{\sigma}_\varepsilon^{-1} \sum_{t=1}^{[T\lambda]} e_t, \tag{5}$$

and where $e_t := y_t - x_t'\hat{\beta}$ are the OLS-residuals from (1), $\hat{\beta} = (\sum_{t=1}^{T} x_t x_t')^{-1}(\sum_{t=1}^{T} x_t y_t)$ is the OLS estimator, and $\hat{\sigma} := (\sum e_t^2 / T)^{1/2}$.

The limiting null distribution of TS is well known for white noise and short memory disturbances. Our first theorem extends these results to stationary long-memory disturbances, where the ε_t follow a stationary ARFIMA(*p*,*d*,*q*) process, i.e., $(1-B)\ \varepsilon_t$ is a stationary ARMA(*p*,*q*) process, and where stationarity requires that $0 < \mathrm{d} < 1/2$. For such processes, it is well known that autocorrelations decay only hyperbolically according to

$$E(\varepsilon_t \varepsilon_{t-k}) = L(k)k^{-d}, \quad k \to \infty \tag{6}$$

where $L(k)$ is slowly varying at infinity.

In the following, "$\Rightarrow$" denotes weak convergence and "$\xrightarrow{P}$" denotes convergence in probability.

Theorem 1**:** In the regression model (1) with disturbance as in (6), we have

$$T^{-d} C_T(\lambda) \Rightarrow B_d(\lambda) - c' Q^{-1} \xi(\lambda), \tag{7}$$

where $B_d(\lambda)$ is fractional Brownian Motion with self-similarity parameter *d* and $\xi(\lambda) \sim N(0,\ \lambda\ \sigma_\varepsilon^2 Q)$.

Remark: Fractional Brownian motion is defined as a Gaussian process with stationary increments and variance $EB_d^2(\lambda) = k\lambda^{2d+1}$, where *k* is a positive constant. It is self-similar with parameter *d*. This means that for every factor *c*, $c^{-d} B_d(c\lambda)$ has the same distribution as the original process $B_d(\lambda)$. The standard Brownian motion is self-similar with parameter 1/2. Fractional Brownian motion has the following integral representation which can be found, for example, in Tsay and Chung (2000):

$$B_d(\lambda) = \sqrt{\frac{(1+2d)\Gamma(1-d)}{\Gamma(1+d)\Gamma(1-2d)}} \times (\int_0^\lambda (\lambda - s) dB(s) + \int_{-\infty}^0 [(\lambda - s)^d - (-\lambda)^d] dB(s)),$$

where *B*(s) denotes standard Brownian motion and Γ(s) denotes the Gamma function.

Proof**:** We have

$$C_T(\lambda) = T^{-\frac{1}{2}}\hat{\sigma}_\varepsilon^{-1}\{\sum_{t=1}^{[T\lambda]}\varepsilon_t - \sum_{t=1}^{[T\lambda]}x_t'(\sum_{t=1}^{T}x_t x_x')^{-1}\sum_{t=1}^{T}x_t\varepsilon_t\}, \tag{8}$$

where $[T\lambda]$ is the largest integer smaller than $T\lambda$. This implies that

$$T^{-d}C_T(\lambda) = \{T^{-d-\frac{1}{2}}z[T\lambda] - T^{-d-\frac{1}{2}}\sum_{t=1}^{[T\lambda]}x_t'(\sum_{1}^{T}x_t x_x')^{-1}\sum_{1}^{T}x_t\varepsilon_t\}/\hat{\sigma}_\varepsilon , \tag{9}$$

where $z_t = z_{t-1} + \varepsilon_t$, $z_0 = 0$ and where the limit in distribution of this expression is to be derived. To see this limit, note that

$$T^{-d-1/2}Z_{[T\lambda]} \Rightarrow \sigma_\varepsilon B_d(\lambda), \tag{10}$$

$$\frac{1}{T}\sum_{t=1}^{[T\lambda]}x_t \to \lambda c, \tag{11}$$

$$(\frac{1}{T}\sum_{t=1}^{[T\lambda]}x_t x_t')^{-1} \to \lambda^{-1}Q^{-1}, \text{ and} \tag{12}$$

$$T^{-d-\frac{1}{2}}\sum_{t=1}^{[T\lambda]}x_t\varepsilon_t \Rightarrow \xi(\lambda). \tag{13}$$

The last convergence follows from a limit theorem of Taqqu (1975), where it is shown that $T^{-d-1/2}\sum_{t=1}^{[T\lambda]}\varepsilon_t$ converges to a Gaussian random variable. Because we assume non-stochastic regressors x_t, this result can be also applied in our situation. The covariance structure of the Gaussian random variable $\xi(\lambda)$ follows by direct algebra and mean zero follows because ε_t was assumed to be a mean zero random variable.

As to the variance, it is easily seen that

$$\hat{\sigma}_\varepsilon^2 = \sum_{t=1}^{T}\frac{e_t 2}{T} = \sum_{t=1}^{T}\frac{\varepsilon 2}{T} + 0p(1) \xrightarrow{P} \sigma_\varepsilon^2. \tag{14}$$

Therefore, collecting together the limits in (10) through (14), the limiting relationship (7) follows.

From (7), it is immediately seen that $TS \xrightarrow{P} \infty$ under H_0, so the OLS-based CUSUM-test is extremely non-robust to long-memory disturbances in the sense that long memory is easily mistaken for structural change when conventional critical values are employed. The convergence to infinity of the test statistics becomes faster for larger values of *d*. Our Monte Carlo experiments in section 3 below will show that this leads to quite dramatic errors of the first kind even for rather small samples and small values of the long-memory parameter *d*.

Next we consider the standard CUSUM-test based on recursive residuals

$$\tilde{e}_t = \frac{y_t - x'\hat{\beta}_t^{(t-1)}}{f_t}, \text{ where}$$

$$\hat{\beta}^{(t-1)} = (X^{(t-1)\prime} X^{t-1})^{-1} X^{(t-1)\prime} y^{(t-1)}, \text{ and} \tag{15}$$

$$f_t = (1 + x_t'(X^{(t-1)\prime} X^{(t-1)})^{-1} x_t)^{\frac{1}{2}} \qquad t = K+1, K+2, \ldots, T \tag{16}$$

and where K is the number of regressors and the superscript $t-1$ means that only observations 1, 2,…, $t-1$ are used. The standard CUSUM-test rejects for large values of

$$S_T = \sup_{0<\lambda<1} W_T(\lambda)/(1+2\lambda), \tag{17}$$

where

$$W_T(\lambda) := T^{-\frac{1}{2}} \hat{\sigma}_\varepsilon^{-1} \sum_{t=K+1}^{[K+\lambda(T-K)]} \tilde{e}_t \quad . \tag{18}$$

Our next result shows that this process likewise converges to a well-defined limit process only after additional normalization:

***Theorem 2*:** In the regression model (1) with disturbances as in (6), we have

$$T^{-d} W_T(\lambda) \rightarrow B_d(\lambda), \tag{19}$$

where again $B_d(\lambda)$ is fractional Brownian Motion with self-similarity parameter d.

***Proof*:** Following Krämer et al. (1988), we write $W_T(\lambda)$ as

$$W_T(\lambda) = \frac{1}{\sqrt{T}} \sum_{t=K+1}^{[K+\lambda(T-K)]} \varepsilon_t - \sum_{t=K+1}^{[K+\lambda(T-K)]} (\hat{\beta}^{(t-1)} - \beta)' x_t. \tag{20}$$

Let $Q_j := 1/T \sum_{i=1}^{j} x_i x_i'$. First we show that

$$\max_{K \le t \le T} \left\| (\hat{\beta}^{(t)} - \beta) - \sum_{j=K}^{t} [(y_j - x_j' \beta) x_j] Q_j^{-1} \right\| = o_p(T^{d+\frac{1}{2}})(\ln \ln T)^{\frac{1}{2}}. \tag{21}$$

Let $S_t := \sum_{j=1}^{t} (y_j - x_j' \beta) x_j$. By the law of the iterated logarithm for the sums of long-memory Gaussian random variables [see Taqqu (1977)] we obtain some bounds for the maximum of S_t. We have for some slowly varying function $L(T)$

$$\max_{1 \le t \le T} \frac{S_t}{[(2/(d+\frac{1}{2})^2(d+\frac{1}{2})-1)T^{2(d+\frac{1}{2})} L(T) lnln(T)]^{\frac{1}{2}}} = O_p(1), \tag{22}$$

so (22) follows directly from Lemma 3.1 of Jureckova and Sen (1984).

Thus we can replace the recursive least squares estimator in (21) by the approximation in (22). Combining (21) and (22) gives

$$\frac{1}{\sqrt{T}}\sum_{t=k+1}^{[T\lambda]}(y_t - x_t'\hat{\beta}^{(t-1)}) = \frac{1}{\sqrt{T}}\sum_{t=1}^{[T\lambda]}\sum_{j=1}^{t} c_{ij}(y_j - x_j'\beta) + o(T^{(d+\frac{1}{2})}\ln\ln T)^{\frac{1}{2}}, \tag{23}$$

where

$$c_{ij} = \begin{cases} -x_j' Q_{(i-1)}^{-1} x_i & i > j \\ 1 & i = j \\ 0 & i < j \end{cases} \tag{24}$$

[see also Sibbertsen (2000)]. In view of a result by Sen (1984), we can now see that the variance of the expression (23) converges to 1. We have

$$\sum_{j \le i} c_{ij}^2 = 1 + O(\frac{1}{i}) . \tag{25}$$

So from theorem 5.1 of Taqqu (1975), one can see as in Theorem 1 that $T^{-d}\omega_T(\lambda)$ converges to fractional Brownian motion. The statement of the theorem is then an immediate consequence of (23).

Theorem 2 shows that the null distribution of the standard CUSUM-test tends to infinity as well, so the standard CUSUM-test has likewise an asymptotic size of unity. Next we check via Monte Carlo how serious this problem is in finite samples.

3. Some Finite Sample Monte Carlo Evidence

To see how far the empirical rejection rates overstate the nominal rejection rates, we generate data according to (1), with disturbances generated via the R statistical software package, which is available free of charge from the internet under http://cran.r-project.org. For the generation of the long-memory disturbances, the routine fracdiff was applied.

Figure 1 below gives the empirical rejection rates using 1000 runs and standard critical values from the i.i.d. disturbance case for the OLS-based CUSUM-test with a nominal significance level of α= 5 when the disturbances are in fact ARFIMA(0,*d*,0). It confirms our theoretical results: rejection rates increase with *d* and sample size and produce misleading evidence even for small *d* and *T*. For instance, for a sample size of T = 200 and d = 0, 2, the true rejection probability is roughly 50% when in fact the test is done under the assumption that it is only 5%.

Figure 2 gives the corresponding empirical rejection rates for the standard CUSUM-test. Not surprisingly, and confirming the intuition which has led us to consider the standard CUSUM-test in the first place, the empirical size is not as far off the mark as for the OLS-based CUSUM-test, but the test is misleading here as well. For instance, for a sample size of T = 200 and d = 0, 2, we have a true rejection rate of about 20%, which is still four times as large as the nominal rejection rate of 5%.

Fig. 1.

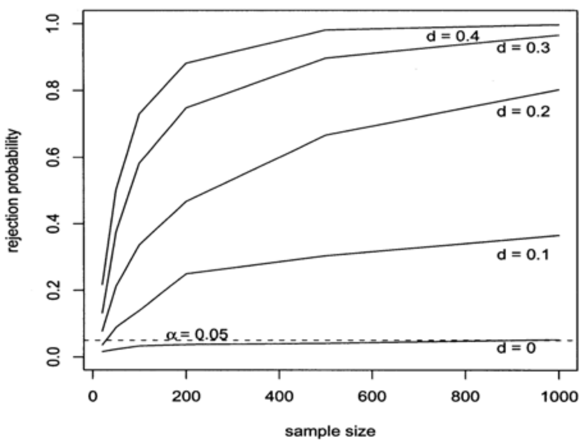
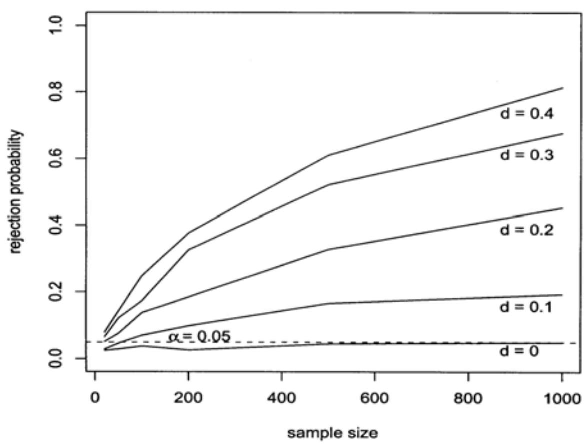


Fig. 2.



Summing up, the paper shows that long-memory disturbances can easily induce the appearance of a structural break, confirming Nunes et al. (1995), Kuan and Hsu (1998), and Hsu (2001), where it was shown that the least squares change point estimator tends to identify a spurious change when there is only long memory.

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